

Extensions of vector-valued functions with preservation of derivatives^{☆,☆☆}M. Koc^{a,*}, Jan Kolář^{b,c}^aRSJ a.s., Na Florenci 2116/15, 110 00 Praha 1, Czech Republic^bInstitute of Mathematics, Czech Academy of Sciences, Žitná 25, 115 67 Praha 1, Czech Republic^cMathematics Institute, University of Warwick, Coventry, UK (September 2014 – October 2015)

Abstract

Let X and Y be Banach or normed linear spaces and $F \subset X$ a closed set. We apply our recent extension theorem for vector-valued Baire one functions to obtain an extension theorem for vector-valued functions $f: F \rightarrow Y$ with pre-assigned derivatives, with preservation of differentiability (at every point where the pre-assigned derivative is actually a derivative), preservation of continuity, preservation of (point-wise) Lipschitz property and (for finite dimensional domain X) preservation of strict differentiability and global (eventually local) Lipschitz continuity. This work depends on the paper *Extensions of vector-valued Baire one functions with preservation of points of continuity* (M. Koc & J. Kolář, J. Math. Anal. Appl. 442:1).

Keywords: vector-valued differentiable functions, extensions, strict differentiability, partitions of unity

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Our results can be roughly viewed as a joint generalization of extension theorems of Tietze-Dugundji, Whitney (C^1 -case) and McShane-Johnson-Lindenstrauss-Schechtman (see [JLS, Theorem 2]), all in point-wise fashion. Nevertheless, they were created as vector-valued generalizations of extension theorems of Aversa, Laczkovich, Preiss [ALP] and Koc, Zajíček [KZ].

1. Introduction

Differentiable extensions of functions were considered already in the 1920's. In [J], V. Jarník proved that every real-valued differentiable function defined on a perfect subset of \mathbb{R} can be extended to an everywhere differentiable function on \mathbb{R} (he even proved a stronger form of this result with preservation of Dini derivatives). This result was independently obtained by G. Petruska and M. Laczkovich in [PL] (even with some additional estimates for the derivative of the extended function) and generalized to real-valued differentiable functions defined on arbitrary closed subsets of \mathbb{R} by J. Mařík in [M]. Extensions of vector-valued functions defined on (not necessarily closed) subsets of \mathbb{R} that preserve the derivative or even some other local properties (e.g. boundedness, continuity or Lipschitz property) were investigated by A. Nekvinda and L. Zajíček in [NZ].

In [ALP, Theorem 7], V. Aversa, M. Laczkovich and D. Preiss proved a result concerning the extendibility to a real-valued differentiable function on \mathbb{R}^n . In particular, they proved that given a function f (defined on some nonempty closed set $F \subset \mathbb{R}^n$) and a derivative of f (with respect to F), there exists an everywhere differentiable extension to \mathbb{R}^n that preserves the prescribed derivative if and only if this prescribed derivative is a Baire one function on F . (Recall that a function is *Baire one* if it is the point-wise limit of a sequence of continuous functions.)

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The existence of continuously differentiable extensions of real-valued functions defined on closed subsets of \mathbb{R}^n was studied in [W] by H. Whitney already in the 1930's (even with preservation of higher orders of smoothness). The vector-valued case can be found, e.g., in [Fe, Theorem 3.1.14].

In [KZ], M. Koc and L. Zajíček proved a result that naturally jointly generalized both the extension result of V. Aversa, M. Laczkovich and D. Preiss [ALP] as well as the C^1 case of the Whitney's extension theorem for real-valued functions defined on closed subsets of \mathbb{R}^n (see, e.g., [EG, § 6.5]). Their result [KZ, Theorem 3.1] can be roughly described as a theorem on extendibility to a differentiable function with preservation of points of continuity of the derivative. We were able to generalize this result further, with the main focus on *vector-valued* functions. We added several other new features, for example non-restrictive assumptions allowing arbitrary function (existing differentiability and continuity points are preserved), the preservation of point-wise, local and global Lipschitz property, or generalization to infinite-dimensional domains. One of the main contributions (extensions of vector-valued Baire one functions) was, due to its different nature and technical difficulty, moved to a separate paper [KK].

Our main results on differentiable extensions (see Theorem 3.1 and Theorem 4.1) can be jointly formulated in the following way (recall that for $p \in \mathbb{N} \cup \{\infty\}$, C^p denotes the class of p -times continuously differentiable functions in Fréchet sense; note that the notion does not change if Fréchet sense is replaced by Gâteaux sense):

Theorem 1.1. *Let X, Y be normed linear spaces, $F \subset X$ a closed set, $f: F \rightarrow Y$ an arbitrary function and $L: F \rightarrow \mathcal{L}(X, Y)$ a Baire one function. Let $p \in \mathbb{N} \cup \{\infty\}$. Then there exists a function $\bar{f}: X \rightarrow Y$ such that*

- (i) $\bar{f} = f$ on F ,
- (ii) if $a \in F$ and f is continuous at a (with respect to F), then \bar{f} is continuous at a ,
- (iii) if $a \in F$, $\alpha \in (0, 1]$ and f is α -Hölder continuous at a (with respect to F), then \bar{f} is α -Hölder continuous at a ; in particular, if f is Lipschitz at a (with respect to F), then \bar{f} is Lipschitz at a ,
- (iv) if $a \in F$ and $L(a)$ is a relative Fréchet derivative of f at a (with respect to F), then $(\bar{f})'(a) = L(a)$,
- (v) \bar{f} is continuous on $X \setminus F$,
- (vi) if X admits C^p -smooth partition of unity, then $\bar{f} \in C^p(X \setminus F, Y)$.

Moreover, if $\dim X < \infty$, then

- (vii) if $a \in F$, L is continuous at a and $L(a)$ is a relative strict derivative of f at a (with respect to F), then the Fréchet derivative $(\bar{f})'$ is continuous at a with respect to $(X \setminus F) \cup \{a\}$ and $L(a)$ is the strict derivative of \bar{f} at a (with respect to X),
- (viii) if $a \in F$, $R > 0$, L is bounded on $B(a, R) \cap F$ and f is Lipschitz continuous on $B(a, R) \cap F$, then \bar{f} is Lipschitz continuous on $B(a, r)$ for every $r < R$; if L is bounded on F and f is Lipschitz continuous on F , then \bar{f} is Lipschitz continuous on X .

Remark 1.2.

- (a) Statement (vi) of Theorem 1.1 can be modified as follows: if X admits \mathcal{F} -partition of unity where \mathcal{F} is a fixed class of functions on X from Lemma 2.5, then $\bar{f}|_{X \setminus F}$ is of class \mathcal{F} (where we say that $g|_U$ is of class \mathcal{F} , on an open set U , if for every $x \in U$, there is a neighborhood V of x such that $g|_V$ is a restriction of a function from \mathcal{F}).
- (b) The assumptions on partitions of unity cannot be removed from (vi), see Proposition 3.3.
- (c) In (vii), if we additionally assume that $L(b)$ is a relative Fréchet derivative of f at b (with respect to F) for every $b \in U$ where U is a relative neighborhood of a in F , then $(\bar{f})'$ is continuous at a (with respect to X).
- (d) The condition $\dim X < \infty$ cannot be removed from (vii) or (viii), see Remark 4.7 for an example.

If the Nagata dimension $\dim_N F$ of F is finite (see [BB1, Definition 4.26] or [BB2, p. 13]), there is no obvious obstacle for extending with preservation of the Lipschitz property [BB2, Theorem 6.26].¹ Thus we rise a natural question:

Question 1.3. Can the condition $\dim X < \infty$ in Theorem 1.1 ((vii) and (viii)) be replaced by condition $\dim_N F < \infty$?

¹Space Y can be used as the range, see [BB2, Proposition 6.5] whose proof works for normed linear spaces.

Remark 1.4. Let F be a closed subset of \mathbb{R}^n , Y be normed linear space. The reader might wonder if, for every function $f: F \rightarrow Y$ differentiable at every point of F (with respect to F), there is a differentiable extension $\tilde{f}: \mathbb{R}^n \rightarrow Y$. The answer depends on quality of F and kinds of differentiability under consideration:

- a) A condition on F that assures the existence of a differentiable extension for every differentiable function f on F can be found in [KZ, Corollary 4.3]. It originates from [ALP, Theorem 4 (ii)]; in both papers, it was formulated with real-valued functions in mind only but it works in the vector-valued case too. Indeed, for F satisfying this condition, the relative derivative $L := f'$ is always a Baire one function on F by [ALP, Proposition 3 (ii), Theorem 4 (ii)] (their proofs work for vector-valued functions as well). For F satisfying the condition, the extension \tilde{f} can be obtained by Theorem 1.1.
- b) However, this extension does not necessarily exist for a general set F : [ALP, Theorem 5] gives an example of a compact set $F \subset \mathbb{R}^2$ and a (uniquely) differentiable function f on F such that f' is not Baire one on F and f therefore cannot be extended to a differentiable function on \mathbb{R}^2 .
- c) A weaker condition on F (namely that $\text{span Tan}(F, x) = \mathbb{R}^n$ for every $x \in \text{der } F$) is sufficient if we ask for a differentiable extension of a *strictly* differentiable function (see [KZ, Proposition 4.10]). This result for real functions can be generalized to vector-valued functions. For more details, see Proposition B.3, where the condition $\text{span Tan}(F, x) = \mathbb{R}^n$ is relaxed to $\text{span Ptg}(F, x) = \mathbb{R}^n$.
- d) For a positive result on C^1 extensions of strictly differentiable functions see [KZ, Corollary 4.7] which can be extended for vector-valued functions with the use of Theorem 1.1, otherwise following the proofs from [KZ]. Again, a condition has to be imposed on the set F , see [KZ, Example 4.14].
- e) Proposition B.4 contains another result on C^1 extensions of strictly differentiable functions, with assumption of continuity of the derivative and, again, a condition on the set F .
- f) For more on this topic see [Ko].

Remark 1.5. Note that all Hilbert spaces and spaces $c_0(\Gamma)$ with arbitrary set Γ admit C^∞ -smooth partition of unity, all reflexive spaces and spaces with a separable dual admit C^1 -smooth partition of unity, L_p spaces with $p \in [1, \infty)$ admit partition of unity of the same smoothness order as their canonical norms. The existence of a C^p -smooth bump implies the existence of C^p -smooth partitions of unity in all separable spaces as well as in all reflexive spaces.

We refer the reader to Remark 2.6 and Remark 2.7 for more information concerning the existence of partitions of unity in various spaces.

The key ingredient in our proof of Theorem 1.1 is our recently developed extension theorem for vector-valued Baire one functions (cf. [ALP, Theorem 6] and [KZ, Theorem 2.4], where special cases of this result were proved):

Theorem 1.6. [KK, Theorem 1.1] *Let (X, ϱ) be a metric space, $F \subset X$ a closed set, Z a normed linear space and $L: F \rightarrow Z$ a Baire one function. Then there exists a continuous function $A: (X \setminus F) \rightarrow Z$ such that*

$$\lim_{\substack{x \rightarrow a \\ x \in X \setminus F}} \|A(x) - L(a)\|_Z \frac{\text{dist}(x, F)}{\varrho(x, a)} = 0 \quad (\text{NT})$$

for every $a \in \partial F$,

$$\lim_{\substack{x \rightarrow a \\ x \in X \setminus F}} A(x) = L(a) \quad (\text{C})$$

whenever $a \in \partial F$ and L is continuous at a , and

$$A \text{ is bounded on } B(a, r) \setminus F \quad (\text{B})$$

whenever $a \in F$, $r \in (0, \infty) \cup \{\infty\}$ and L is bounded on $B(a, 12r) \cap F$.

Besides this introductory section, our paper consists of three more sections. Section 2 contains the definitions of the notions related to derivatives and partitions of unity as well as an auxiliary proposition about relativizations of partitions of unity to open subsets. Section 3 is devoted to extensions of vector-valued functions from closed subsets of *infinite* dimensional spaces, whereas Section 4 extends the proofs of the previous section to obtain stronger results for *finite* dimensional domains. There is also technical Appendix A on partitions of unity and Appendix B containing a small elaboration of a theme from [KZ, Section 4].

2. Basic notions and preliminaries

Let X and Y be normed linear spaces. We denote by $\mathcal{L}(X, Y)$ the set of all bounded linear operators from X to Y . For $u \in \mathcal{L}(X, Y)$, the number

$$\|u\|_{\mathcal{L}(X, Y)} := \inf \{K > 0 : \|u(x)\|_Y \leq K \|x\|_X \text{ for every } x \in X\}$$

is called the *norm* of the linear operator u . The set $\mathcal{L}(X, Y)$ equipped with the norm $\|\cdot\|_{\mathcal{L}(X, Y)}$ forms a normed linear space, which is complete provided Y is complete.

Definition 2.1. Let X and Y be normed linear spaces, $A \subset X$ an arbitrary set, $f: A \rightarrow Y$ a function and $a \in A$.

- (i) A bounded linear operator $F_a: X \rightarrow Y$ is called a *relative Fréchet derivative of f at a* (with respect to A) if either a is an isolated point of A , or

$$\lim_{\substack{x \rightarrow a \\ x \in A}} \frac{\|f(x) - f(a) - F_a(x - a)\|_Y}{\|x - a\|_X} = 0. \quad (1)$$

- (ii) A bounded linear operator $S_a: X \rightarrow Y$ is called a *relative strict derivative of f at a* (with respect to A) if either a is an isolated point of A , or

$$\lim_{\substack{y \rightarrow a \\ x \rightarrow a \\ x, y \in A, x \neq y}} \frac{\|f(y) - f(x) - S_a(y - x)\|_Y}{\|y - x\|_X} = 0 \quad (\text{with } x = a \text{ or } y = a \text{ allowed}). \quad (2)$$

- (iii) We say that $L: A \rightarrow \mathcal{L}(X, Y)$ is a *relative Fréchet* (resp. *strict*) *derivative of f (on A)* if $L(a)$ is a relative Fréchet (resp. strict) derivative of f at a (with respect to A) for each $a \in A$.

Remark 2.2.

- (a) At interior points of A , the notion of relative Fréchet (resp. strict) derivative agrees with the classical notion of the Fréchet (resp. strict) derivative. In particular, if it exists then it is uniquely determined.
- (b) Classically, Fréchet differentiability of functions between normed linear spaces is introduced only for functions defined on open sets.
- (c) A relative strict derivative of f is clearly also a relative Fréchet derivative of f .
- (d) If we consider $X = \mathbb{R}^n$ and a function $f: A \subset \mathbb{R}^n \rightarrow Y$ in Definition 2.1, then we can omit the assumption of boundedness for operators F_a and S_a , since every linear function defined on a finite-dimensional normed linear space is bounded automatically.
- (e) If $X = \mathbb{R}^n$ then F_a is called a *relative derivative of f at a* or, if a is an interior point of A , the *derivative of f at a* .

Definition 2.3. Let X and Y be normed linear spaces, $A \subset X$ an arbitrary set, $f: A \rightarrow Y$ a function and $a \in A$.

If $\alpha \in (0, 1]$, we say that f is *α -Hölder continuous at a* (with respect to A) if either a is an isolated point of A , or

$$\limsup_{\substack{x \rightarrow a \\ x \in A}} \frac{\|f(x) - f(a)\|_Y}{\|x - a\|_X^\alpha} < \infty.$$

We say that f is *Lipschitz at a* (with respect to A) if it is 1-Hölder continuous at a (with respect to A), i.e., either a is an isolated point of A , or

$$\limsup_{\substack{x \rightarrow a \\ x \in A}} \frac{\|f(x) - f(a)\|_Y}{\|x - a\|_X} < \infty.$$

As usual, f is *point-wise α -Hölder* (*point-wise Lipschitz*) if f is α -Hölder continuous (Lipschitz) at every $a \in A$. And f is *locally α -Hölder* (*locally Lipschitz*) if it is α -Hölder (Lipschitz) (in the classical sense) in a neighborhood of every $a \in A$.

If $U = X$ then the following definitions agree with the usual notions (see [KM, 16.1., p. 165] or [HHZ, p. 304]). If $U \subset X$ is a general open set, we essentially consider the restrictions to U .

Definition 2.4. Let X be a metric space, \mathcal{F} a class of functions on X and $U \subset X$ an open set.

A *locally finite partition of unity in U* (shortly a *partition of unity in U*) is a collection $\{\psi_\gamma\}_{\gamma \in \Gamma}$ of real-valued functions ψ_γ on X such that $\sum_{\gamma \in \Gamma} \psi_\gamma(x) = 1$ for every $x \in U$ and there is a neighborhood V_γ of γ , for every $\gamma \in \Gamma$, so that all but a finite number of ψ_γ vanish on V_γ .

If $\psi_\gamma \in \mathcal{F}$ for every $\gamma \in \Gamma$, we talk about an \mathcal{F} -*partition of unity*. If \mathcal{F} is not specified, usually the continuous functions are assumed.

We say that a (locally finite) partition of unity $\{\psi_\gamma\}_{\gamma \in \Gamma}$ in U is *subordinated to* an open cover \mathcal{U} of U if for every $\gamma \in \Gamma$ there is $U_\gamma \in \mathcal{U}$ such that $\text{supp}(\psi_\gamma) \subset U_\gamma$, where $\text{supp}(\psi_\gamma) = \overline{\{x \in X : \psi_\gamma(x) \neq 0\}}$.

We say that U *admits \mathcal{F} -partition of unity* if for every open cover \mathcal{U} of U there is a locally finite \mathcal{F} -partition of unity $\{\psi_\gamma\}_{\gamma \in \Gamma}$ in U subordinated to \mathcal{U} .

Lemma 2.5. Let X be a normed linear space. Let \mathcal{F} be

- (a) the class of all continuous functions on X , or
- (b) the class of all continuous functions on X that are p_1 -times Gâteaux differentiable for some $p_1 \in \mathbb{N} \cup \{\infty\}$, or
- (c) the class of all p_2 -times Fréchet differentiable functions on X for some $p_2 \in \mathbb{N} \cup \{\infty\}$, or
- (d) the class of all C^{p_3} -smooth functions on X for some $p_3 \in \mathbb{N} \cup \{\infty\}$, or
- (e) the class of all point-wise α_1 -Hölder continuous functions on X for some $\alpha_1 \in (0, 1]$, or
- (f) the class of all locally α_2 -Hölder continuous functions on X for some $\alpha_2 \in (0, 1]$, in particular the class of all locally Lipschitz continuous functions on X , or
- (g) the intersection of two or several of the above classes (for some $p_1, p_2, p_3, \alpha_1, \alpha_2$).

Let \mathcal{F}^+ be the class of all non-negative functions from \mathcal{F} . If X admits \mathcal{F} -partition of unity, then every open set $U \subset X$ admits \mathcal{F}^+ -partition of unity.

Proof. We get immediately that X admits \mathcal{F}^+ -partition of unity. Indeed, given a locally finite partition of unity $\{\psi_\gamma\}_{\gamma \in \Gamma} \subset \mathcal{F}$, we put $\tilde{\psi}_\gamma = \psi_\gamma^2 / \sum_{\beta \in \Gamma} \psi_\beta^2$ for every $\gamma \in \Gamma$. Then $\sum_{\gamma \in \Gamma} \tilde{\psi}_\gamma = 1$ and, for every $\gamma \in \Gamma$, $\tilde{\psi}_\gamma \in \mathcal{F}^+$ and $\text{supp } \tilde{\psi}_\gamma = \text{supp } \psi_\gamma$.

Let $U \subset X$ be an arbitrary open set and let \mathcal{U} be an open cover of U . Set $F = X \setminus U$. Define

$$d_n = \begin{cases} 1/n, & n \in \mathbb{N}, \\ \infty, & n \leq 0. \end{cases}$$

Fix $n \in \mathbb{N}$. Let

$$\begin{aligned} U_n &= \{x \in X : d_n < \text{dist}(x, F) < d_{n-3}\}, \\ F_n &= \{x \in X : d_{n-1} \leq \text{dist}(x, F) \leq d_{n-2}\}, \\ \mathcal{U}_n &= \{U_n \cap G : G \in \mathcal{U}\} \cup \{X \setminus F_n\}. \end{aligned}$$

Then \mathcal{U}_n is an open cover of X . Let $\{\phi_{n,\gamma}\}_{\gamma \in \mathcal{A}_n}$ be an \mathcal{F}^+ -partition of unity subordinated to \mathcal{U}_n . Let $\mathcal{B}_n = \{\gamma \in \mathcal{A}_n : \text{supp } \phi_{n,\gamma} \not\subset X \setminus F_n\}$. Then $\{\phi_{n,\gamma}\}_{\gamma \in \mathcal{B}_n}$ is subordinated to \mathcal{U} , $\sum_{\gamma \in \mathcal{B}_n} \phi_{n,\gamma}(x) = 1$ for every $x \in F_n$ and $\text{supp } \phi_{n,\gamma} \subset U_n$ for every $\gamma \in \mathcal{B}_n$.

The family $\{\phi_{n,\gamma} : n \in \mathbb{N}, \gamma \in \mathcal{B}_n\}$ is subordinated to \mathcal{U} and locally finite in U . Every $x \in U$ belongs to one or two of the sets F_n , and at most three sets U_n . Thus

$$1 \leq w(x) := \sum_{n \in \mathbb{N}, \gamma \in \mathcal{B}_n} \phi_{n,\gamma} \leq 3 \quad (3)$$

for every $x \in U$. For every $n \in \mathbb{N}$ and $\gamma \in \mathcal{B}_n$, let $\psi_{n,\gamma}(x) = \phi_{n,\gamma}(x)/w(x)$ for $x \in U$ and note again that the sum in (3) is finite in a neighborhood of every point of $U \supset U_n \supset \text{supp } \phi_{n,\gamma}$. For every $n \in \mathbb{N}$ and $\gamma \in \mathcal{B}_n$, extend $\psi_{n,\gamma}$ by setting $\psi_{n,\gamma}(x) = 0$ for $x \in X \setminus U$. Then $\{\psi_{n,\gamma}\}_{n \in \mathbb{N}, \gamma \in \mathcal{B}_n}$ is a locally finite \mathcal{F}^+ -partition of unity in U . \square

Remark 2.6.

- (a) Every metric space admits partition of unity formed by continuous functions. Moreover, it even admits partition of unity formed by Lipschitz continuous (hence locally Lipschitz continuous, as used in Lemma 2.5) functions (see [Fr, the proof of Theorem]).
- (b) If X is a WCD Banach space, then X admits partition of unity formed by continuous functions that are Gâteaux differentiable [DGZ2, Corollary VIII.3.3]. The class of WCD spaces contains all separable and all reflexive Banach spaces [DGZ2, Example VI.2.2].
- (c) There is a Banach space that is not WCD and admits C^∞ -smooth partition of unity, e.g. JL space of W.B. Johnson and J. Lindenstrauss (see [DGZ2, p. 369], for the definition of JL space, see [JL1]).
- (d) If X^* is a WCG Banach space, then X admits C^1 -smooth partition of unity [DGZ2, Corollary VIII.3.11]. This includes all reflexive spaces as well as all spaces with a separable dual.
- (e) If a Banach space X admits a LUR norm whose dual norm is also LUR, then X admits C^1 -smooth partition of unity [DGZ2, Theorem VIII.3.12 (i)]. Hence, if $K^{(\omega_1)} = \emptyset$, then $C(K)$ admits C^1 -smooth partition of unity [DGZ2, Corollary VIII.3.13]. Note that if $K^{(\omega_0)} = \emptyset$, then $C(K)$ admits even C^∞ -smooth partition of unity (see [DGZ1]).
- (f) L_p spaces with $p \in [1, \infty)$ admit partition of unity of the same smoothness order as their canonical norms, i.e. C^∞ -smooth partition of unity for p even integer, C^{p-1} -smooth partition of unity for p odd integer and, if p is not an integer, $C^{[p]}$ -smooth partition of unity, where $[p]$ denotes the integer part of p (see [DGZ2, Corollary VIII.3.11] and [DGZ2, Theorem V.1.1]).
- (g) All Hilbert spaces and spaces $c_0(\Gamma)$ with arbitrary set Γ admit C^∞ -smooth partition of unity [T, Theorem 2 and Theorem 3] (see also [KM, 16.16]).

Remark 2.7. Let $p \in \mathbb{N} \cup \{\infty\}$.

- (a) It is an open problem whether every Banach space that admits a C^p -smooth bump must also admit C^p -smooth partition of unity (see [DGZ2, p. 370, Problem VIII.1], [FM, p. 179] and [KM, p. 172]).
- (b) The existence of a C^p -smooth bump implies the existence of C^p -smooth partitions of unity for example for separable spaces (see [BF] or [DGZ2, p. 360]) and for reflexive spaces [DGZ2, Theorem VIII.3.2].
More generally, it also holds for Banach spaces whose dual is WCG [KM, 16.13(4)], for WCD Banach spaces [DGZ2, p. 351, Theorem VIII.3.2] (cf. also [KM, 53.15 and 16.18]), which includes reflexive spaces and separable spaces as we already noted, and for duals of Asplund spaces [KM, 53.15 and 16.18].
A result on Banach spaces with PRI and C^p -smooth partitions of unity can be found in [H, Corollary 4]. In particular, Banach spaces with "nice" ("separable") PRI and with a C^p -smooth bump function admit C^p -smooth partition of unity, see [GTWZ, Remark 3.3], [DGZ2, page 369, lines 26–27] and [KM, 16.18].

3. Vector-valued functions in infinite dimensional domain

Theorem 3.1. *Let X, Y be normed linear spaces, $F \subset X$ a closed set, $f: F \rightarrow Y$ an arbitrary function and $L: F \rightarrow \mathcal{L}(X, Y)$ a function that is Baire one on F . Then there exists a function $\bar{f}: X \rightarrow Y$ such that*

- (i) $\bar{f} = f$ on F ,
- (ii) if $a \in F$ and f is continuous at a (with respect to F), then \bar{f} is continuous at a ,
- (iii) if $a \in F$, $\alpha \in (0, 1]$ and f is α -Hölder continuous at a (with respect to F), then \bar{f} is α -Hölder continuous at a ; in particular, if f is Lipschitz at a (with respect to F), then \bar{f} is Lipschitz at a ,
- (iv) if $a \in F$ and $L(a)$ is a relative Fréchet derivative of f at a (with respect to F), then $(\bar{f})'(a) = L(a)$,
- (v) \bar{f} is continuous on $X \setminus F$,
- (vi) if X admits \mathcal{F} -partition of unity where \mathcal{F} is a fixed class of functions on X from Lemma 2.5, then $\bar{f}|_{X \setminus F}$ is of class \mathcal{F} .²

²To provide a formal definition for (vi), we say that $g|_U$ is of class \mathcal{F} (on an open set U) if for every $x \in U$, there is a neighborhood V of x such that $g|_V$ is a restriction of a function from \mathcal{F} .

Proof. If $F = \emptyset$, the theorem trivially holds. Further suppose that F is nonempty. For every $x \in X$, we set

$$r(x) := \frac{1}{20} \text{dist}(x, F). \quad (4)$$

Further, for every $x \in X \setminus F$, we choose any point $\widehat{x} \in F$ such that

$$\|x - \widehat{x}\|_X \leq 2 \text{dist}(x, F). \quad (5)$$

If (vi) is under consideration, X admits \mathcal{F} -partition of unity. If this is not the case, it admits at least continuous partition of unity (since X is a metric space) and we let \mathcal{F} be the class of continuous functions on X .

By Lemma 2.5, there exists a non-negative locally finite \mathcal{F} -partition of unity $\{\phi_\gamma\}_{\gamma \in \Gamma}$ on $X \setminus F$ subordinated to the covering $\{B(x, 10r(x)) : x \in X \setminus F\}$. So, in particular,

$$\{\phi_\gamma\}_{\gamma \in \Gamma} \subset \mathcal{F}, \quad (6)$$

$$0 \leq \phi_\gamma \text{ for every } \gamma \in \Gamma, \quad (7)$$

$$\sum_{\gamma \in \Gamma} \phi_\gamma(x) = 1 \text{ for every } x \in X \setminus F \quad (8)$$

and for every $\gamma \in \Gamma$ there is $x_\gamma \in X \setminus F$ such that

$$\text{supp } \phi_\gamma \subset B(x_\gamma, 10r(x_\gamma)). \quad (9)$$

For every $x \in X \setminus F$, we denote

$$\Gamma_x := \{\gamma \in \Gamma : B(x, 10r(x)) \cap B(x_\gamma, 10r(x_\gamma)) \neq \emptyset\}. \quad (10)$$

Clearly, if $\gamma \in \Gamma \setminus \Gamma_x$ then $\phi_\gamma(x) = 0$ by (9). Moreover, if $\gamma \in \Gamma_x$ then

$$|r(x) - r(x_\gamma)| \leq \text{Lip}(r) \|x - x_\gamma\|_X = \frac{1}{20} \|x - x_\gamma\|_X \stackrel{(10)}{\leq} \frac{1}{20} (10r(x) + 10r(x_\gamma)).$$

Hence

$$\frac{1}{3} \leq \frac{r(x)}{r(x_\gamma)} \leq 3 \quad \text{whenever } \gamma \in \Gamma_x. \quad (11)$$

Let $A : (X \setminus F) \rightarrow \mathcal{L}(X, Y)$ be the function constructed in Theorem 1.6 (with $Z = \mathcal{L}(X, Y)$).

Define $\tilde{f} : X \rightarrow Y$ by

$$\tilde{f}(x) := \begin{cases} f(x) & \text{if } x \in F, \\ \sum_{\gamma \in \Gamma} \phi_\gamma(x) [f(\widehat{x}_\gamma) + A(x_\gamma)(x - \widehat{x}_\gamma)] & \text{if } x \in X \setminus F. \end{cases} \quad (12)$$

Obviously, $\tilde{f} = f$ on F , which proves (i).

Since linear mappings are C^∞ -smooth and the partition of unity $\{\phi_\gamma\}_{\gamma \in \Gamma}$ is locally finite, we easily conclude using (6) that $\tilde{f}|_{X \setminus F}$ is of class \mathcal{F} . Assertions (vi), if under consideration, and (v) are therefore fulfilled.

Let $a \in F$. For arbitrary $x \in X \setminus F$ and $\gamma \in \Gamma_x$, by (4), (5), (10) and (11), we get

$$\|x_\gamma - x\|_X \leq 10r(x_\gamma) + 10r(x) \leq 40r(x) = 2 \text{dist}(x, F), \quad (13)$$

and likewise with x_γ in the place of x on the right-hand side

$$\|x_\gamma - x\|_X \leq 10r(x_\gamma) + 10r(x) \leq 40r(x_\gamma) = 2 \text{dist}(x_\gamma, F), \quad (14)$$

$$\|\widehat{x} - x_\gamma\|_X \leq \|\widehat{x} - x\|_X + \|x - x_\gamma\|_X \leq 2 \text{dist}(x, F) + 2 \text{dist}(x, F) = 4 \text{dist}(x, F), \quad (15)$$

$$\|\widehat{x}_\gamma - x_\gamma\|_X \leq 2 \text{dist}(x_\gamma, F) \leq 2 \|\widehat{x} - x_\gamma\|_X \leq 8 \text{dist}(x, F),$$

$$\|\widehat{x}_\gamma - \widehat{x}\|_X \leq \|\widehat{x}_\gamma - x_\gamma\|_X + \|x_\gamma - \widehat{x}\|_X \leq 8 \operatorname{dist}(x, F) + 4 \operatorname{dist}(x, F) = 12 \operatorname{dist}(x, F), \quad (16)$$

$$\|\widehat{x}_\gamma - x\|_X \leq \|\widehat{x}_\gamma - x_\gamma\|_X + \|x_\gamma - x\|_X \leq 8 \operatorname{dist}(x, F) + 2 \operatorname{dist}(x, F) = 10 \operatorname{dist}(x, F), \quad (17)$$

and likewise

$$\|\widehat{x}_\gamma - x\|_X \leq \|\widehat{x}_\gamma - x_\gamma\|_X + \|x_\gamma - x\|_X \leq 2 \operatorname{dist}(x_\gamma, F) + 2 \operatorname{dist}(x_\gamma, F) = 4 \operatorname{dist}(x_\gamma, F). \quad (18)$$

Since $\operatorname{dist}(x, F) \leq \|x - a\|_X$, by (5), (13) and (17), we obtain

$$\|x_\gamma - a\|_X \leq \|x_\gamma - x\|_X + \|x - a\|_X \leq 3 \|x - a\|_X, \quad (19)$$

$$\|\widehat{x}_\gamma - a\|_X \leq \|\widehat{x}_\gamma - x\|_X + \|x - a\|_X \leq 11 \|x - a\|_X, \quad (20)$$

$$\|\widehat{x} - a\|_X \leq \|\widehat{x} - x\|_X + \|x - a\|_X \leq 3 \|x - a\|_X. \quad (21)$$

Since assertions (ii), (iii) and (iv) are clearly satisfied for $a \in \operatorname{int}(F)$, we will further assume that $a \in \partial F$. If $x \in X \setminus F$, by (7), (8), (9), (10) and (12), we obtain

$$\begin{aligned} \|\bar{f}(x) - \bar{f}(a)\|_Y &= \left\| \sum_{\gamma \in \Gamma} \phi_\gamma(x) [f(\widehat{x}_\gamma) + A(x_\gamma)(x - \widehat{x}_\gamma) - f(a)] \right\|_Y \\ &= \left\| \sum_{\gamma \in \Gamma} \phi_\gamma(x) [f(\widehat{x}_\gamma) - f(a) + L(a)(x - \widehat{x}_\gamma) + (A(x_\gamma) - L(a))(x - \widehat{x}_\gamma)] \right\|_Y \\ &\leq \sum_{\gamma \in \Gamma_x} \phi_\gamma(x) \|f(\widehat{x}_\gamma) - f(a)\|_Y + \sum_{\gamma \in \Gamma_x} \phi_\gamma(x) \|L(a)\|_{\mathcal{L}(X,Y)} \|x - \widehat{x}_\gamma\|_X \\ &\quad + \sum_{\gamma \in \Gamma_x} \phi_\gamma(x) \|A(x_\gamma) - L(a)\|_{\mathcal{L}(X,Y)} \operatorname{dist}(x_\gamma, F) \frac{\|x - \widehat{x}_\gamma\|_X}{\operatorname{dist}(x_\gamma, F)}. \end{aligned} \quad (22)$$

First suppose that f is continuous at a (with respect to F) and fix $\varepsilon_1 > 0$. There exists $\delta_1 > 0$ such that

$$\|f(z) - f(a)\|_Y \leq \varepsilon_1 \quad \text{for every } z \in F, \|z - a\|_X < \delta_1. \quad (23)$$

By (NT) from Theorem 1.6, there exists $\delta_2 > 0$ such that

$$\|A(t) - L(a)\|_{\mathcal{L}(X,Y)} \operatorname{dist}(t, F) < \varepsilon_1 \quad \text{for every } t \in X \setminus F, \|t - a\|_X < \delta_2. \quad (24)$$

Let $x \in X \setminus F$ be arbitrary with $\|x - a\|_X < \min(\varepsilon_1, \frac{\delta_1}{11}, \frac{\delta_2}{3})$. Then we deduce from (19) and (20) that $\|x_\gamma - a\|_X < \delta_2$ and $\|\widehat{x}_\gamma - a\|_X < \delta_1$ for every $\gamma \in \Gamma_x$. Thus by (7), (8), (17), (18), (22), (23), (24) and $\operatorname{dist}(x, F) \leq \|x - a\|_X$, we obtain

$$\|\bar{f}(x) - \bar{f}(a)\|_Y \leq \varepsilon_1 + 10 \|L(a)\|_{\mathcal{L}(X,Y)} \varepsilon_1 + 4 \varepsilon_1 = (5 + 10 \|L(a)\|_{\mathcal{L}(X,Y)}) \varepsilon_1.$$

Since $\varepsilon_1 > 0$ was arbitrary, \bar{f} is continuous at a and thus (ii) is proved.

Next, suppose that $\alpha \in (0, 1]$ and f is α -Hölder continuous at a (with respect to F). Then there exist $K > 0$ and $\delta_3 > 0$ such that

$$\|f(z) - f(a)\|_Y \leq K \|z - a\|_X^\alpha \quad \text{for every } z \in F, \|z - a\|_X < \delta_3. \quad (25)$$

By (NT) from Theorem 1.6, there exists $\delta_4 > 0$ such that

$$\|A(t) - L(a)\|_{\mathcal{L}(X,Y)} \operatorname{dist}(t, F) < \|t - a\|_X \quad \text{for every } t \in X \setminus F, \|t - a\|_X < \delta_4. \quad (26)$$

Let $x \in X \setminus F$ such that $\|x - a\|_X < \min(\frac{\delta_3}{11}, \frac{\delta_4}{3}, 1)$. Then, for every $\gamma \in \Gamma_x$, using (19) and (20) we get $\|x_\gamma - a\|_X < \delta_4$ and $\|\widehat{x}_\gamma - a\|_X < \delta_3$. Similarly as above, by (7), (8), (17), (18), (19), (20), (22), (25), (26) and $\operatorname{dist}(x, F) \leq \|x - a\|_X$,

we get

$$\begin{aligned} \|\bar{f}(x) - \bar{f}(a)\|_Y &\leq K \sum_{\gamma \in \Gamma_x} \phi_\gamma(x) \|\widehat{x}_\gamma - a\|_X^\alpha + 10 \|L(a)\|_{\mathcal{L}(X,Y)} \|x - a\|_X \\ &\quad + 4 \sum_{\gamma \in \Gamma_x} \phi_\gamma(x) \|x_\gamma - a\|_X \\ &\leq (11^\alpha K + 10 \|L(a)\|_{\mathcal{L}(X,Y)} + 12) \|x - a\|_X^\alpha, \end{aligned} \quad (27)$$

since $\|x - a\|_X \leq \|x - a\|_X^\alpha$ as $\|x - a\|_X < 1$ and $\alpha \in (0, 1]$. Hence \bar{f} is α -Hölder continuous at a and (iii) is proved.

Finally, we prove (iv). Fix $\varepsilon_2 > 0$. Since $L(a)$ is a Fréchet derivative of f at a (with respect to F), there exists $\delta_5 > 0$ such that

$$\|f(z) - f(a) - L(a)(z - a)\|_Y \leq \varepsilon_2 \|z - a\|_X \quad \text{for every } z \in F, \|z - a\|_X < \delta_5. \quad (28)$$

By (NT) from Theorem 1.6, there exists $\delta_6 > 0$ such that

$$\|A(t) - L(a)\|_{\mathcal{L}(X,Y)} \frac{\text{dist}(t, F)}{\|t - a\|_X} < \varepsilon_2 \quad \text{for every } t \in X \setminus F, \|t - a\|_X < \delta_6. \quad (29)$$

Let $x \in X \setminus F$ be arbitrary satisfying $\|x - a\|_X < \min(\frac{\delta_5}{11}, \frac{\delta_6}{3})$. Then, for every $\gamma \in \Gamma_x$, we get $\|x_\gamma - a\|_X < \delta_6$ and $\|\widehat{x}_\gamma - a\|_X < \delta_5$ by (19) and (20). Thus by (7), (8), (9), (10), (12), (18), (19), (20), (28) and (29), we obtain

$$\begin{aligned} \|\bar{f}(x) - \bar{f}(a) - L(a)(x - a)\|_Y &= \left\| \sum_{\gamma \in \Gamma} \phi_\gamma(x) [f(\widehat{x}_\gamma) + A(x_\gamma)(x - \widehat{x}_\gamma) - f(a) - L(a)(x - a)] \right\|_Y \\ &= \left\| \sum_{\gamma \in \Gamma} \phi_\gamma(x) [f(\widehat{x}_\gamma) - f(a) - L(a)(\widehat{x}_\gamma - a) + (A(x_\gamma) - L(a))(x - \widehat{x}_\gamma)] \right\|_Y \\ &\leq \sum_{\gamma \in \Gamma_x} \phi_\gamma(x) \|f(\widehat{x}_\gamma) - f(a) - L(a)(\widehat{x}_\gamma - a)\|_Y \\ &\quad + \sum_{\gamma \in \Gamma_x} \phi_\gamma(x) \|A(x_\gamma) - L(a)\|_{\mathcal{L}(X,Y)} \|x - \widehat{x}_\gamma\|_X \\ &\leq \sum_{\gamma \in \Gamma_x} \phi_\gamma(x) \varepsilon_2 \|\widehat{x}_\gamma - a\|_X \\ &\quad + \sum_{\gamma \in \Gamma_x} \phi_\gamma(x) \|A(x_\gamma) - L(a)\|_{\mathcal{L}(X,Y)} \frac{\text{dist}(x_\gamma, F)}{\|x_\gamma - a\|_X} \frac{\|x - \widehat{x}_\gamma\|_X}{\text{dist}(x_\gamma, F)} \|x_\gamma - a\|_X \\ &\leq 11 \varepsilon_2 \|x - a\|_X + 12 \varepsilon_2 \|x - a\|_X = 23 \varepsilon_2 \|x - a\|_X. \end{aligned}$$

Since $\varepsilon_2 > 0$ was arbitrary, we finally get

$$\lim_{\substack{x \rightarrow a \\ x \in X \setminus F}} \frac{\|\bar{f}(x) - \bar{f}(a) - L(a)(x - a)\|_Y}{\|x - a\|_X} = 0.$$

Since $L(a)$ is a Fréchet derivative of \bar{f} at a with respect to F , we deduce $(\bar{f})'(a) = L(a)$, which proves (iv). \square

By a straightforward application of Theorem 3.1, we obtain the following generalization of [ALP, Theorem 7] for infinite-dimensional domains and vector-valued functions.

Corollary 3.2. *Let X be a normed linear space that admits Fréchet differentiable partition of unity, $F \subset X$ a nonempty closed set, Y a normed linear space, $f: F \rightarrow Y$ an arbitrary function and $L: F \rightarrow \mathcal{L}(X, Y)$ a relative Fréchet derivative of f (with respect to F). Then L is Baire one on F if and only if there exists a function $\bar{f}: X \rightarrow Y$ such that \bar{f} extends f , \bar{f} is Fréchet differentiable everywhere on X and $(\bar{f})' = L$ on F .*

The following proposition shows that the assumption on partitions of unity cannot be removed from (vi). The remaining statements of Theorem 3.1 require only continuous partitions of unity which are available in all metric spaces.

Proposition 3.3. *Let X be a normed linear space and $\tilde{p} \in \mathbb{N} \cup \{\infty\}$. The following statements are equivalent:*

- (a) *The space X admits $C^{\tilde{p}}$ -smooth partition of unity or partition of unity formed by continuous functions \tilde{p} -times differentiable in Fréchet or Gâteaux sense.*
- (b) *For every normed linear space Y , a nonempty closed set $F \subset X$, a function $f: F \rightarrow Y$ and a Baire one function $L: F \rightarrow \mathcal{L}(X, Y)$, there exists a function $\tilde{f}: X \rightarrow Y$ that satisfies the conclusions of Theorem 3.1 including the respective conclusion of (vi) with $p = \tilde{p}$.*
- (c) *Given any nonempty closed set $F \subset X$ and a Fréchet smooth (or even locally constant) function $f: F \rightarrow \mathbb{R}$, there exists a function $\tilde{f}: X \rightarrow \mathbb{R}$ that satisfies at least the following properties from Theorem 3.1: (i), (ii) and the respective conclusion of (vi) with $p = \tilde{p}$.*

Proof. The implication (b) \Rightarrow (c) is obvious and (a) \Rightarrow (b) follows by Theorem 3.1. The third implication (c) \Rightarrow (a) follows by [DGZ2, Lemma VIII.3.6, (ii) \Rightarrow (i)] (or, more precisely, the proof of it, since the argument does not use the completeness of X) as soon as we show that given sets $A \subset W \subset X$, where A is closed and W open, there exists a $C^{\tilde{p}}$ -smooth function $h: X \rightarrow [0, 1]$ such that $A \subset h^{-1}(0, \infty) \subset W$. To do so, assume A and W are as indicated. Set $B = X \setminus W$ and $F = A \cup B$. Let $L(x) = 0 \in X^*$ for $x \in F$ and $f(x) = 1$ for $x \in A$, $f(x) = 0$ for $x \in B$. By (c), there exists a function \tilde{f} that satisfies conclusions (i), (ii) and (vi) of Theorem 3.1 (with $p = \tilde{p}$). This extension \tilde{f} is not necessarily \tilde{p} -times continuously differentiable on the boundary of F . However, $h(x) := \varphi(\tilde{f}(x))$ satisfies all required properties if φ is a suitable smooth function (e.g., $\varphi: \mathbb{R} \rightarrow [0, 1]$ with $\varphi = 0$ on $(-\infty, 1/4]$ and $\varphi = 1$ on $[3/4, \infty)$); h' vanishes in a neighborhood of the boundary of F by (i) and (ii). \square

4. Vector-valued functions in finite dimensional domain

In this section, the domain space is the Euclidean space \mathbb{R}^n ($n \in \mathbb{N}$). The norm on \mathbb{R}^n is denoted by $|\cdot|$. We identify \mathbb{R}^n with its dual space $(\mathbb{R}^n)^*$ of all linear functionals on \mathbb{R}^n .

It will be convenient to use the following *tensor product* notation. If $\psi \in X^*$ and $y \in Y$, then $(y \otimes \psi)(u) := \psi(u)y$ for every $u \in X$. Note that $y \otimes \psi \in \mathcal{L}(X, Y)$. In particular, if $\phi: \mathbb{R}^n \rightarrow \mathbb{R}$ is differentiable at $x \in \mathbb{R}^n$ and $y \in Y$, then $y \otimes \phi'(x) \in \mathcal{L}(\mathbb{R}^n, Y)$ and $(y \otimes \phi'(x))(u) = (\phi'(x))(u)y \in Y$ for every $u \in \mathbb{R}^n$. Hence $y \otimes \phi'(x)$ is the derivative of vector-valued function $t \mapsto \phi(t)y$ at x .

The following theorem generalizes the main extension result from [KZ] to the case of vector-valued functions (see Corollary 4.4, compare with [KZ, Theorem 3.1]).

Theorem 4.1. *Let $F \subset \mathbb{R}^n$ be a closed set, Y a normed linear space, $f: F \rightarrow Y$ an arbitrary function and $L: F \rightarrow \mathcal{L}(\mathbb{R}^n, Y)$ a function that is Baire one on F . Then there exists a function $\tilde{f}: \mathbb{R}^n \rightarrow Y$ such that*

- (i) $\tilde{f} = f$ on F ,
- (ii) *if $a \in F$ and f is continuous at a (with respect to F), then \tilde{f} is continuous at a ,*
- (iii) *if $a \in F$, $\alpha \in (0, 1]$ and f is α -Hölder continuous at a (with respect to F), then \tilde{f} is α -Hölder continuous at a ; in particular, if f is Lipschitz at a (with respect to F), then \tilde{f} is Lipschitz at a ,*
- (iv) *if $a \in F$ and $L(a)$ is a relative Fréchet derivative of f at a (with respect to F), then $(\tilde{f})'(a) = L(a)$,*
- (v) $\tilde{f}|_{\mathbb{R}^n \setminus F} \in C^\infty(\mathbb{R}^n \setminus F, Y)$,
- (vi) *if $a \in F$, L is continuous at a and $L(a)$ is a relative strict derivative of f at a (with respect to F), then the Fréchet derivative $(\tilde{f})'$ is continuous at a with respect to $(\mathbb{R}^n \setminus F) \cup \{a\}$ and $L(a)$ is the strict derivative of \tilde{f} at a (with respect to \mathbb{R}^n),*
- (vii) *if $a \in F$, $R > 0$, L is bounded on $B(a, R) \cap F$ and f is Lipschitz continuous on $B(a, R) \cap F$, then \tilde{f} is Lipschitz continuous on $B(a, r)$ for every $r < R$; if L is bounded on F and f is Lipschitz continuous on F , then \tilde{f} is Lipschitz continuous on \mathbb{R}^n .*

The strategy of the proof is analogous to the one used in the proof of Theorem 3.1. Assertions (i)-(v) follow directly from Theorem 3.1 as \mathbb{R}^n admits C^∞ -smooth partition of unity. To ensure (vi)-(vii), we need a special C^∞ -smooth partition of unity in $\mathbb{R}^n \setminus F$ that meets several additional requirements analogous to those used in proofs of [KZ, Theorem 3.1] and the C^1 case of Whitney's extension theorem in [EG], namely (32) and (38) below. Since we decided to include the preservation of the global Lipschitz continuity (see (vii)), we had to introduce a slight change compared to [KZ] and [EG].

Lemma 4.2. *There are $C_1, C_2 > 1$ depending only on the dimension $n \in \mathbb{N}$ with the following property: Let $F \subset \mathbb{R}^n$ be a nonempty closed set. There exist $\{x_j\}_{j \in \mathbb{N}} \subset \mathbb{R}^n \setminus F$ and $\{\phi_j\}_{j \in \mathbb{N}} \subset C^\infty(\mathbb{R}^n \setminus F, \mathbb{R})$ such that, letting*

$$\mathcal{J}_x := \{j \in \mathbb{N} : B(x, 10r(x)) \cap B(x_j, 10r(x_j)) \neq \emptyset\} \quad (30)$$

and

$$r(x) := \frac{1}{20} \text{dist}(x, F), \quad (31)$$

we have, for every $j \in \mathbb{N}$ and $x \in \mathbb{R}^n \setminus F$,

$$\text{Card}(\mathcal{J}_x) \leq C_1, \quad (32)$$

$$\frac{1}{3} \leq \frac{r(x)}{r(x_j)} \leq 3 \quad \text{if } j \in \mathcal{J}_x, \quad (33)$$

$$0 \leq \phi_j, \quad (34)$$

$$\text{supp } \phi_j \subset B(x_j, 10r(x_j)), \quad (35)$$

$$\sum_{j \in \mathbb{N}} \phi_j(x) = 1, \quad (36)$$

$$\sum_{j \in \mathbb{N}} \phi_j'(x) = 0 \quad (37)$$

and

$$|\phi_j'(x)| \leq \frac{C_2}{r(x)}. \quad (38)$$

The proof of Lemma 4.2 is standard. It can be derived from a very similar statement that is proven in [EG, pp. 245–247] and summarized in [KZ, Step 1 on p. 1031]. Statements in the same spirit can also be found in [S] and [G, Theorem 2.2]. For the sake of completeness, we prove the lemma in Appendix A.

Proof of Theorem 4.1. If F is empty, the theorem trivially holds. Further suppose that F is nonempty. Let $C_1, C_2 > 1$, $\{x_j\}_{j \in \mathbb{N}} \subset \mathbb{R}^n \setminus F$, $\{\phi_j\}_{j \in \mathbb{N}} \subset C^\infty(\mathbb{R}^n \setminus F, \mathbb{R})$, \mathcal{J}_x and $r(x)$ be as in Lemma 4.2. For every $x \in \mathbb{R}^n \setminus F$, we choose any point $\widehat{x} \in F$ such that

$$|x - \widehat{x}| = \text{dist}(x, F). \quad (39)$$

Let $A : (\mathbb{R}^n \setminus F) \rightarrow \mathcal{L}(\mathbb{R}^n, Y)$ be the function constructed in Theorem 1.6 (with $X = \mathbb{R}^n$ and $Z = \mathcal{L}(\mathbb{R}^n, Y)$). Define $\tilde{f} : \mathbb{R}^n \rightarrow Y$ by

$$\tilde{f}(x) := \begin{cases} f(x) & \text{if } x \in F, \\ \sum_{j \in \mathbb{N}} \phi_j(x) [f(\widehat{x}_j) + A(x_j)(x - \widehat{x}_j)] & \text{if } x \in \mathbb{R}^n \setminus F. \end{cases} \quad (40)$$

As the formula for the extended function \tilde{f} is the same one as in the proof of Theorem 3.1 and the partition of unity $\{\phi_j\}_{j \in \mathbb{N}}$ in $\mathbb{R}^n \setminus F$ is only a special case of the partition of unity $\{\phi_\gamma\}_{\gamma \in \Gamma}$ used in the proof of Theorem 3.1, assertions (i)-(v) follow immediately by applying the proof of Theorem 3.1 for the special case when $X = \mathbb{R}^n$.

It remains to prove assertions (vi)-(vii). We need some auxiliary estimates and computations. Let $a \in F$. For arbitrary $x \in \mathbb{R}^n \setminus F$ and $j \in \mathcal{J}_x$, by (33), (30), (31) and (39), we get

$$|x_j - x| \leq 10r(x_j) + 10r(x) \leq 40r(x) = 2 \operatorname{dist}(x, F), \quad (41)$$

and likewise with x_j in the place of x on the right-hand side

$$|x_j - x| \leq 10r(x_j) + 10r(x) \leq 40r(x_j) = 2 \operatorname{dist}(x_j, F), \quad (42)$$

$$|\widehat{x} - x_j| \leq |\widehat{x} - x| + |x - x_j| \leq \operatorname{dist}(x, F) + 2 \operatorname{dist}(x, F) = 3 \operatorname{dist}(x, F), \quad (43)$$

$$|\widehat{x}_j - x_j| = \operatorname{dist}(x_j, F) \leq |\widehat{x} - x_j| \leq 3 \operatorname{dist}(x, F),$$

$$|\widehat{x}_j - \widehat{x}| \leq |\widehat{x}_j - x_j| + |x_j - \widehat{x}| \leq 3 \operatorname{dist}(x, F) + 3 \operatorname{dist}(x, F) = 6 \operatorname{dist}(x, F), \quad (44)$$

$$|\widehat{x}_j - x| \leq |\widehat{x}_j - x_j| + |x_j - x| \leq 3 \operatorname{dist}(x, F) + 2 \operatorname{dist}(x, F) = 5 \operatorname{dist}(x, F). \quad (45)$$

Since $\operatorname{dist}(x, F) \leq |x - a|$, by (39), (41) and (45), we obtain

$$|x_j - a| \leq |x_j - x| + |x - a| \leq 3|x - a|, \quad (46)$$

$$|\widehat{x}_j - a| \leq |\widehat{x}_j - x| + |x - a| \leq 6|x - a|, \quad (47)$$

$$|\widehat{x} - a| \leq |\widehat{x} - x| + |x - a| \leq 2|x - a|. \quad (48)$$

For $x \in \mathbb{R}^n \setminus F$, differentiating \bar{f} at x , by (32), (35), (36), (37), (30) and (40), we get

$$\begin{aligned} (\bar{f})'(x) &= \sum_{j \in \mathcal{J}_x} \phi_j(x) A(x_j) + \sum_{j \in \mathcal{J}_x} [f(\widehat{x}_j) + A(x_j)(x - \widehat{x}_j)] \otimes \phi_j'(x) \\ &= \sum_{j \in \mathcal{J}_x} \phi_j(x) L(a) + \sum_{j \in \mathcal{J}_x} \phi_j(x) [A(x_j) - L(a)] \\ &\quad + \sum_{j \in \mathcal{J}_x} [f(\widehat{x}) - L(a)(\widehat{x} - x)] \otimes \phi_j'(x) \\ &\quad + \sum_{j \in \mathcal{J}_x} [f(\widehat{x}_j) - f(\widehat{x}) - L(a)(\widehat{x}_j - \widehat{x})] \otimes \phi_j'(x) \\ &\quad + \sum_{j \in \mathcal{J}_x} [(A(x_j) - L(a))(x - \widehat{x}_j)] \otimes \phi_j'(x) \\ &= L(a) + \sum_{j \in \mathcal{J}_x} \phi_j(x) [A(x_j) - L(a)] \\ &\quad + \sum_{j \in \mathcal{J}_x} [f(\widehat{x}_j) - f(\widehat{x}) - L(a)(\widehat{x}_j - \widehat{x})] \otimes \phi_j'(x) \\ &\quad + \sum_{j \in \mathcal{J}_x} [(A(x_j) - L(a))(x - \widehat{x}_j)] \otimes \phi_j'(x). \end{aligned} \quad (49)$$

Now, we direct our attention to assertions (vi) and (vii).

Claim 4.3. *Let the following be defined as above: $F \subset \mathbb{R}^n$, Y a normed linear space, $L: F \rightarrow \mathcal{L}(\mathbb{R}^n, Y)$, $f: F \rightarrow Y$, $\bar{f}: \mathbb{R}^n \rightarrow Y$ and $A: (\mathbb{R}^n \setminus F) \rightarrow \mathcal{L}(\mathbb{R}^n, Y)$. Suppose that $a \in F$, $r_1, r_2 \in (0, \infty) \cup \{\infty\}$ and $K_1, K_2 \geq 0$ satisfy*

$$\|A(t) - L(a)\|_{\mathcal{L}(\mathbb{R}^n, Y)} \leq K_1 \quad \text{for every } t \in \mathbb{R}^n \setminus F, |t - a| < r_1 \quad (50)$$

and

$$\|f(z) - f(y) - L(a)(z - y)\|_Y \leq K_2|z - y| \quad \text{for every } y, z \in F, \max(|y - a|, |z - a|) < r_2. \quad (51)$$

For $x, y \in \mathbb{R}^n$, denote

$$E_{xy} := \|\bar{f}(y) - \bar{f}(x) - L(a)(y - x)\|_Y = \sup_{\substack{T \in Y^* \\ \|T\|_{Y^*} \leq 1}} \left| T(\bar{f}(y) - \bar{f}(x) - L(a)(y - x)) \right|. \quad (52)$$

Let $r_3 = \min(r_1/3, r_2/6)$ and $K_3 = (1 + 5 \cdot 20C_1C_2)K_1 + 6 \cdot 20C_1C_2K_2$, where C_1, C_2 are the constants from Lemma 4.2. Then

$$\|(\bar{f})'(x) - L(a)\|_{\mathcal{L}(\mathbb{R}^n, Y)} \leq K_3 \quad \text{for all } x \in \mathbb{R}^n \setminus F \text{ such that } |x - a| < r_3 \quad (53)$$

and

$$E_{xy} \leq 33K_3 |y - x| \quad \text{for all } x, y \in \mathbb{R}^n \text{ such that } \max(|x - a|, |y - a|) < r_3/2. \quad (54)$$

Postponing the proof of Claim 4.3, we now proceed to the proof of assertion (vi). As its conclusion clearly holds for $a \in \text{int}(F)$, we can further assume that $a \in \partial F$. Fix $\varepsilon_1 > 0$. Note that L is assumed to be continuous at a (with respect to F). By (C) from Theorem 1.6, there exists $r_1 > 0$ such that (cf. (50))

$$\|A(t) - L(a)\|_{\mathcal{L}(\mathbb{R}^n, Y)} < \varepsilon_1 \quad \text{for every } t \in \mathbb{R}^n \setminus F, |t - a| < r_1. \quad (55)$$

Since we assume that $L(a)$ is a strict derivative of f at a (with respect to F), there exists $r_2 > 0$ such that (cf. (51))

$$\|f(z) - f(y) - L(a)(z - y)\|_Y \leq \varepsilon_1 |z - y| \quad \text{for every } y, z \in F, \max(|y - a|, |z - a|) < r_2. \quad (56)$$

By (53) from Claim 4.3 applied with $K_1 = K_2 = \varepsilon_1$, we get $r_3 > 0$ such that

$$\|(\bar{f})'(x) - L(a)\|_{\mathcal{L}(\mathbb{R}^n, Y)} \leq K_3 \quad \text{for all } x \in \mathbb{R}^n \setminus F \text{ such that } |x - a| < r_3, \quad (57)$$

with $K_3 = [1 + 220C_1C_2] \varepsilon_1$. Since $\varepsilon_1 > 0$ was arbitrary and $(\bar{f})'(a) = L(a)$ (note that we already proved (iv)), we get that $(\bar{f})'$ is continuous at a with respect to $(\mathbb{R}^n \setminus F) \cup \{a\}$.

Likewise, the estimate of E_{xy} provided by (54) shows that $L(a)$ is the strict derivative of \bar{f} at a . Hence, the proof of assertion (vi) is finished.

To prove assertion (vii), we prove that if $a \in F$, $r \in (0, \infty) \cup \{\infty\}$, L is bounded on $B(a, 72r) \cap F$ and f is Lipschitz continuous on $B(a, 12r) \cap F$, then \bar{f} is Lipschitz continuous on $B(a, r)$. Both statements of (vii) then obviously follow either by a standard compactness argument or using the case $r = \infty$.

Assume that $a \in F$, $r \in (0, \infty) \cup \{\infty\}$, L is bounded on $B(a, 72r) \cap F$ and f is Lipschitz continuous on $B(a, 12r) \cap F$. Let K_0 denote the Lipschitz constant of f . Then there is $C_0 > 0$ such that $\|A\|_{\mathcal{L}(\mathbb{R}^n, Y)} \leq C_0$ on $B(a, 6r) \cap (\mathbb{R}^n \setminus F)$ since A was obtained from Theorem 1.6 (cf. (B)). Thus we have (50) with $K_1 = C_0 + \|L(a)\|_{\mathcal{L}(\mathbb{R}^n, Y)}$ and $r_1 = 6r$. Using the Lipschitz property of f , we obtain (51) with $K_2 = K_0 + \|L(a)\|_{\mathcal{L}(\mathbb{R}^n, Y)}$ and $r_2 = 12r$. An application of Claim 4.3, namely of (54), gives

$$\|\bar{f}(y) - \bar{f}(x)\|_Y \leq (33K_3 + \|L(a)\|_{\mathcal{L}(\mathbb{R}^n, Y)}) |y - x|$$

for every $x, y \in \mathbb{R}^n$ such that $\max(|x - a|, |y - a|) < r_3/2 = r$, which is the required Lipschitz property of \bar{f} , cf. (vii).

This concludes the proof of Theorem 4.1 except that we still have to show that Claim 4.3 holds true. \square

Proof of Claim 4.3. Let also the other symbols be defined as above (that is, x_j, ϕ_j ($j \in \mathbb{N}$), \mathcal{J}_x and $r(x)$ ($x \in \mathbb{R}^n \setminus F$) are as in Lemma 4.2, \widehat{x} as in (39) etc.). Let $x \in \mathbb{R}^n \setminus F$ and $|x - a| < r_3 := \min\left(\frac{r_1}{3}, \frac{r_2}{6}\right)$. Then, for every $j \in \mathcal{J}_x$, using (46), (47) and (48), we get $|x_j - a| < r_1$ and $\max(|\widehat{x}_j - a|, |\widehat{x} - a|) < r_2$.

By (49),

$$\begin{aligned} \|(\bar{f})'(x) - L(a)\|_{\mathcal{L}(\mathbb{R}^n, Y)} &\leq \sum_{j \in \mathcal{J}_x} \phi_j(x) \|A(x_j) - L(a)\|_{\mathcal{L}(\mathbb{R}^n, Y)} \\ &\quad + \sum_{j \in \mathcal{J}_x} \|f(\widehat{x}_j) - f(\widehat{x}) - L(a)(\widehat{x}_j - \widehat{x})\|_Y |\phi_j'(x)| \\ &\quad + \sum_{j \in \mathcal{J}_x} \|A(x_j) - L(a)\|_{\mathcal{L}(\mathbb{R}^n, Y)} |x - \widehat{x}_j| |\phi_j'(x)|. \end{aligned}$$

Estimating the first term by (50) together with (34) and (36), the second one by (51) with (32), (38) and (44), and the third one by (50) with (32), (38) and (45), we get

$$\begin{aligned} \|(\bar{f})'(x) - L(a)\|_{\mathcal{L}(\mathbb{R}^n, Y)} &\leq K_1 + 6 \operatorname{dist}(x, F) \frac{20C_1C_2}{\operatorname{dist}(x, F)} K_2 + 5 \operatorname{dist}(x, F) \frac{20C_1C_2}{\operatorname{dist}(x, F)} K_1 \\ &\leq (1 + 5 \cdot 20C_1C_2)K_1 + 6 \cdot 20C_1C_2K_2 = K_3. \end{aligned} \quad (58)$$

Thus we obtained (53).

Next, we want to prove (54), the estimate of E_{xy} . If (ii) of Theorem 4.1 were applicable at every point of $a \in F$, this could have been done easily using the continuity of $(\bar{f})'$ at $a \in F$ (also the mean value theorem would be used on parts of the segment L_{xy} together with the estimate $E_{xy} \leq E_{xu} + E_{uv} + E_{vy}$ analogously to the arguments that follow), but we can deal with the general case as well.

From (50), we have

$$\|A(x)\|_{\mathcal{L}(\mathbb{R}^n, Y)} \leq M \quad (59)$$

whenever $|x - a| < r_1$, where $M = \|L(a)\|_{\mathcal{L}(\mathbb{R}^n, Y)} + K_1$.

Note that $K_2 \leq K_3$ by the definition of K_3 , since $C_1, C_2 > 1$. Fix $x, y \in \mathbb{R}^n$ such that $\max(|x - a|, |y - a|) < r_3/2$. We will show that

$$E_{xy} \leq 33K_3 |y - x|.$$

As this inequality trivially holds for $x = y$, we will further suppose that $x \neq y$.

Let L_{xy} denote the (closed) segment connecting x and y . We will distinguish several possible cases.

If $L_{xy} \subset \mathbb{R}^n \setminus F$ and $T \in Y^*$ with $\|T\|_{Y^*} \leq 1$, then there exists $\xi_T \in L_{xy}$ such that

$$T(\bar{f}(y) - \bar{f}(x)) = ((T(\bar{f}))'(\xi_T))(y - x).$$

By (58), we simply get

$$E_{xy} \leq \sup_{\substack{T \in Y^* \\ \|T\|_{Y^*} \leq 1}} \|T\|_{Y^*} \|(\bar{f})'(\xi_T) - L(a)\|_{\mathcal{L}(\mathbb{R}^n, Y)} |y - x| \leq K_3 |y - x|. \quad (60)$$

If $x, y \in F$, we have $E_{xy} \leq K_2 |y - x|$ by (51).

In the remaining cases, $L_{xy} \cap F \neq \emptyset$ and one or both points x, y lie in $\mathbb{R}^n \setminus F$.

If $x, y \in \mathbb{R}^n \setminus F$ then segment L_{xy} can be divided into two or three segments as follows:

1. L_{xu} with $u \in F$ and $L_{xu} \setminus \{u\} \subset \mathbb{R}^n \setminus F$.
2. L_{uv} with $u, v \in F$, which might possibly be degenerate ($v = u$).
3. L_{vy} with $v \in F$ and $L_{vy} \setminus \{v\} \subset \mathbb{R}^n \setminus F$.

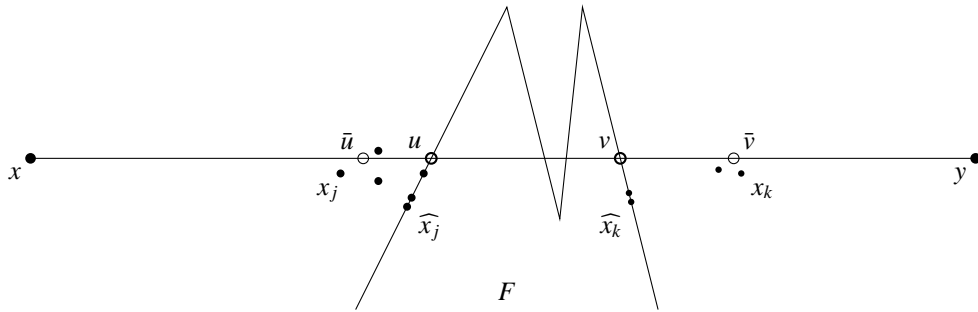


Figure 1: The case $x, y \in \mathbb{R}^n \setminus F$ with L_{xy} intersecting F . Published with permission of © Jan Kolář 2016. All Rights Reserved.

The reader might welcome an informal remark, that we will not use the estimate of E_{uv} (which could be obtained immediately from (51)), but replace it by a convex combination of estimates of $E_{\widehat{x}_j, \widehat{x}_k}$ with $\widehat{x}_j, \widehat{x}_k$ related to the definition of $\bar{f}(\bar{u}), \bar{f}(\bar{v})$, where $\bar{u}, \bar{v} \in \mathbb{R}^n \setminus F$ are points approximating u, v (see Figure 1). This way we do not need the continuity of \bar{f} at points $u, v \in F$.

We omit the case $x \in F, y \in \mathbb{R}^n \setminus F$ since it is analogous to the case that follows.

If

$$x \in \mathbb{R}^n \setminus F \text{ and } y \in F \quad (61)$$

then L_{xy} divides into two segments L_{xu} and L_{uv} as above with $v = y$ (we can consider L_{vy} as degenerate). We use a convex combination of estimates of $E_{\widehat{x}_j, y}$ (again provided by (51)). Apart from that, this case is similar to the most complex case $x, y \in \mathbb{R}^n \setminus F$ and therefore we will not fully treat both of them. (Formally, the case (61) can be treated together with the case $x, y \in \mathbb{R}^n \setminus F$ if we extend our notation as follows: Let $\phi_0(z) = 1$ if $z \in F$ and $\phi_0(z) = 0$ if $z \in \mathbb{R}^n \setminus F$. Let $x_0 = y, \widehat{x}_0 = y$ and $A(x_0) = 0$. Then $\{\phi_j\}_{j \in \mathbb{N} \cup \{0\}}$ is a partition of unity and (40) remains true, with unchanged values of \bar{f} , if the sum is extended to include $j = 0$. Moreover, the second line of (40) then gives the correct value of $\bar{f}(v)$ even though we have $v = y \in F$. We also define $\mathcal{J}_{x_0} = \{0\}$.)

Let us concentrate on the case $x, y \in \mathbb{R}^n \setminus F$. Let

$$m := |y - x| \min(K_3/M, 1/4). \quad (62)$$

We choose a point $\bar{u} \in L_{xu} \setminus \{u\}$ with $|\bar{u} - u| < m$ and likewise $\bar{v} \in L_{vy} \setminus \{v\}$ with $|\bar{v} - v| < m$. (For the case (61) we let $\bar{v} = v = y$.) Since $L_{x\bar{u}} \subset \mathbb{R}^n \setminus F$, we already estimated in (60) that

$$E_{x\bar{u}} \leq K_3 |\bar{u} - x| \leq K_3 |y - x|. \quad (63)$$

Likewise,

$$E_{\bar{v}y} \leq K_3 |y - \bar{v}| \leq K_3 |y - x|. \quad (64)$$

By (45), we have

$$|\widehat{x}_j - \bar{u}| \leq 5 \operatorname{dist}(\bar{u}, F) \leq 5 |\bar{u} - u| < 5m \quad (65)$$

$$\text{and } |\widehat{x}_k - \bar{v}| \leq 5 |\bar{v} - v| < 5m \quad (66)$$

whenever $j \in \mathcal{J}_{\bar{u}}$ and $k \in \mathcal{J}_{\bar{v}}$, in which case therefore also

$$\left| (\widehat{x}_k - \widehat{x}_j) - (\bar{v} - \bar{u}) \right| \leq |\widehat{x}_k - \bar{v}| + |\widehat{x}_j - \bar{u}| \leq 10m. \quad (67)$$

Since $\bar{u} \in L_{xy} \subset B(a, r_3/2)$, clearly $|\bar{u} - a| < r_3/2$, and from (41), we get $|x_j - a| \leq |x_j - \bar{u}| + |\bar{u} - a| \leq 2 \operatorname{dist}(\bar{u}, F) + |\bar{u} - a| \leq 3 |\bar{u} - a| < 3r_3/2 \leq r_1$ whenever $j \in \mathcal{J}_{\bar{u}}$. The values of $\bar{f}(\bar{u})$ (and similarly also of $\bar{f}(\bar{v})$) are defined by (40) where $\phi_j(\bar{u})$ can be nonzero only when $j \in \mathcal{J}_{\bar{u}}$. Using (40), the triangle inequality, (34), (36), (59) and (65), we obtain

$$\left\| \bar{f}(\bar{u}) - \sum_j \phi_j(\bar{u}) f(\widehat{x}_j) \right\|_Y \leq \sum_j \phi_j(\bar{u}) \|A(x_j)\|_{\mathcal{L}(\mathbb{R}^n, Y)} |\bar{u} - \widehat{x}_j| \leq 5Mm \leq 5K_3 |y - x|.$$

Likewise,

$$\left\| \bar{f}(\bar{v}) - \sum_k \phi_k(\bar{v}) f(\widehat{x}_k) \right\|_Y \leq 5Mm \leq 5K_3 |y - x|.$$

Using identities $\phi_j = \phi_j \sum_k \phi_k$ and $\phi_k = \phi_k \sum_j \phi_j$, we can write

$$\left\| \bar{f}(\bar{v}) - \bar{f}(\bar{u}) - \sum_j \sum_k \phi_j(\bar{u}) \phi_k(\bar{v}) (f(\widehat{x}_k) - f(\widehat{x}_j)) \right\|_Y \leq 10K_3 |y - x|. \quad (68)$$

Since $\widehat{x}_j, \widehat{x}_k \in F$, we get by (51), $K_2 \leq K_3$, (67) and (62)

$$\left\| f(\widehat{x}_k) - f(\widehat{x}_j) - L(a)(\widehat{x}_k - \widehat{x}_j) \right\|_Y \leq K_2 |\widehat{x}_k - \widehat{x}_j| \leq K_3 (10m + |\bar{u} - \bar{v}|) \leq 11K_3 |y - x|$$

whenever $j \in \mathcal{J}_{\bar{u}}$ and $k \in \mathcal{J}_{\bar{v}}$. Hence, again by (67), we obtain (see also (62) and note that $\|L(a)\|_{\mathcal{L}(\mathbb{R}^n, Y)} \leq M$)

$$\|f(\bar{x}_k) - f(\bar{x}_j) - L(a)(\bar{v} - \bar{u})\|_Y \leq 11K_3 |y - x| + 10m \|L(a)\|_{\mathcal{L}(\mathbb{R}^n, Y)} \leq 21K_3 |y - x|,$$

which combines with (68) and $\sum_j \sum_k \phi_j(\bar{u}) \phi_k(\bar{v}) = 1$ to

$$\|\bar{f}(\bar{v}) - \bar{f}(\bar{u}) - L(a)(\bar{v} - \bar{u})\|_Y \leq 31K_3 |y - x|. \quad (69)$$

So

$$E_{\bar{u}\bar{v}} \leq 31K_3 |y - x|. \quad (70)$$

By (63), (70) and (64), since $E_{xy} \leq E_{x\bar{u}} + E_{\bar{u}\bar{v}} + E_{\bar{v}y}$,

$$E_{xy} \leq 33K_3 |y - x|, \quad (71)$$

which concludes the proof of Claim 4.3. \square

The following corollary provides a vector-valued version of [KZ, Theorem 3.1].

Corollary 4.4. *Let $F \subset \mathbb{R}^n$ be a nonempty closed set, Y a normed linear space, $f: F \rightarrow Y$ an arbitrary function and $L: F \rightarrow \mathcal{L}(\mathbb{R}^n, Y)$ a relative Fréchet derivative of f (on F) such that L is Baire one on F . Then there exists a function $\bar{f}: \mathbb{R}^n \rightarrow Y$ such that*

- (i) \bar{f} is Fréchet differentiable on \mathbb{R}^n ,
- (ii) $\bar{f} = f$ and $(\bar{f})' = L$ on F ,
- (iii) if $a \in F$, L is continuous at a and $L(a)$ is a relative strict derivative of f at a (with respect to F), then the Fréchet derivative $(\bar{f})'$ is continuous at a ,
- (iv) $\bar{f} \in C^\infty(\mathbb{R}^n \setminus F, Y)$.

Remark 4.5. The previous corollary easily implies the C^1 case of Whitney's extension theorem for vector-valued functions (see, e.g., [Fe, Theorem 3.1.14]). Indeed, assuming that the assumptions of Whitney's theorem are fulfilled, it is sufficient to show that $L(a)$ is a strict derivative of f at a for every $a \in F$ (which involves a straightforward and easy computation only, cf. [KZ, Remark 3.2]) and then to apply Corollary 4.4.

Remark 4.6.

- (a) In (vi) of Theorem 4.1, we cannot expect the Fréchet derivative $(\bar{f})'$ to be continuous at a with respect to the whole space \mathbb{R}^n unless appropriate assumptions are added (cf. Remark 1.2(c)). Indeed, consider $n = 2$, $F = [0, 1] \times \{0\}$, $f: F \rightarrow \mathbb{R}$ given by $f(x, 0) = x^7 \left| \sin \frac{1}{x} \right|$ for $x \in (0, 1]$ and $f(0, 0) = 0$, $L = 0$ and $a = (0, 0)$. Note that $L(a)$ is a relative *strict* derivative of f at a . If we extend f according to Theorem 4.1, then the extended function \bar{f} is not Fréchet differentiable in any neighborhood of a , since both f and \bar{f} do not have a Fréchet derivative at those points of F at which $\sin \frac{1}{x}$ changes its sign. However, the Fréchet derivative $(\bar{f})'$ is continuous at a with respect to $(\mathbb{R}^2 \setminus F) \cup \{a\}$ as Theorem 4.1(vi) states.
- (b) In Theorem 4.1(vi), neither the continuity of $(\bar{f})'$ at a nor the conclusion that $L(a)$ is the strict derivative of \bar{f} at a (even with respect to $(\mathbb{R}^n \setminus F) \cup \{a\}$) can be obtained when we remove the assumption that $L(a)$ is a relative *strict* derivative of f at a (with respect to F). Indeed, consider $F = \{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}\} \subset \mathbb{R}$ and let $f: F \rightarrow \mathbb{R}$ be given by $f(\frac{1}{n}) = \frac{(-1)^n}{n^2}$ for $n \in \mathbb{N}$ and $f(0) = 0$. Let $L(x) = 0$ ($x \in F$) and $a = 0$. Obviously, $L(a) = 0$ is a relative derivative of f at a with respect to F . For $n \in \mathbb{N}$, the distance between $x_n := \frac{1}{n}$ and x_{n+1} is less than $\frac{1}{n^2}$ and the absolute increment of f between these two points is greater than $\frac{1}{n^2}$. Applying Theorem 4.1, we obtain the extended function \bar{f} on \mathbb{R} that is continuous at every x_n due to condition (ii). By the mean value theorem, for every $n \in \mathbb{N}$, the absolute value of the derivative of \bar{f} at some point of the interval (x_{n+1}, x_n) is greater than 1. Therefore $(\bar{f})'$ cannot be continuous at a with respect to $(\mathbb{R} \setminus F) \cup \{a\}$. Also, $L(a) = 0$ cannot be a strict derivative of \bar{f} at a (even with respect to $(\mathbb{R} \setminus F) \cup \{a\}$).

- (c) Likewise, neither of the two conclusions of Theorem 4.1(vi)³ can be obtained without assuming that $L(a)$ is continuous at a with respect to F . Consider the same set F as in (b) together with $a = 0$, $f = 0$ on F , $L(0) = 0$ and $L(\frac{1}{n}) = (-1)^n$ for $n \in \mathbb{N}$. Applying Theorem 4.1, we obtain the extended function \tilde{f} on \mathbb{R} that has $(-1)^n$ as the derivative at isolated point $\frac{1}{n}$ due to condition (iv). Hence \tilde{f} is continuous at $\frac{1}{n}$. By the mean value theorem, for every $n \in \mathbb{N}$, the absolute value of the derivative of \tilde{f} at some point close to $\frac{1}{n}$ is greater than $\frac{1}{2}$. Note that $(\tilde{f})'(0) = L(0) = 0$. Therefore $(\tilde{f})'$ cannot be continuous at a with respect to $(\mathbb{R} \setminus F) \cup \{a\}$. Also, $L(a) = 0$ cannot be a strict derivative of \tilde{f} at a (even with respect to $(\mathbb{R} \setminus F) \cup \{a\}$).

Remark 4.7. If statements (vi) or (vii) are required in Theorem 4.1, the assumption $F \subset \mathbb{R}^n$ cannot be generalized to $F \subset X$, replacing \mathbb{R}^n by an (infinitely dimensional) Banach space X . In other words, the condition $\dim X < \infty$ cannot be removed from Theorem 1.1 (vii)(viii).

Indeed, let $p \in [1, 2)$, $X = L_p(0, 1)$, $Y = l_2$, $e \in X$ with $\|e\|_X = 1$. By [JL2, Theorem 3], for every integer $n > 10$, there is a finite set $F_n \subset X$ and a function $f_n: F_n \rightarrow Y$, such that $\text{Lip } f > c_n \text{Lip } f_n$ for every $f: X \rightarrow Y$ that extends f_n , where $c_n \rightarrow \infty$. The actual value of $c_n = \tau \cdot (\log n / \log \log n)^{1/p-1/2}$ (for some $\tau > 0$) is not important for our purposes. By translating and scaling down the set, and by scaling the values of f_n we can assure that $F_n \subset B_X(2^{-n}e, 2^{-2n-1})$, $f_n(F_n) \subset B_Y(0, 2^{-2n-1})$ and $\text{Lip } f_n = c_n^{-1/2}$. Then the property of f_n is that it has no extension $\tilde{f}_n: X \rightarrow Y$ with $\text{Lip } \tilde{f}_n \leq d_n := c_n \text{Lip } f_n = c_n^{1/2}$. Note that $c_n^{-1/2} \rightarrow 0$ and $d_n \rightarrow \infty$. Let $F = \{0\} \cup \bigcup_{n>10} F_n$ and define $f: F \rightarrow Y$ by $f(0) = 0$, $f|_{F_n} = f_n$. The scaling was chosen so that $0 \in \mathcal{L}(X, Y)$ is a relative strict derivative of f (with respect to F) at $a := 0 \in X$. Since every F_n is finite, 0 is the only accumulation point of F . Let $L(x) = 0$ for $x \in F$. Consider \tilde{f} that is an extension of f as in the theorem.

If $(\tilde{f})'$ is continuous at a with respect to $(X \setminus F) \cup \{a\}$ as in (vi), we obtain a contradiction. First, we see that $(\tilde{f})'$ is actually continuous with respect to X (note that the derivative is continuous at $a = 0$ with respect to F , since it equals L on F because every $x \in F \setminus \{a\}$ is an isolated point of F). Consequently, \tilde{f} is Lipschitz in $B_X(0, 2r)$ for some $r > 0$. Let π be the radial projection onto $\overline{B_X(0, r)}$, $\pi(x) = x$ for $x \in B_X(0, r)$ and $\pi(x) = rx/\|x\|$ for $x \in X \setminus B_X(0, r)$. Then $\text{Lip } \pi \leq 2$, see e.g. [Mal, Remark 4], hence the mapping $g(x) = \tilde{f}(\pi(x))$ is Lipschitz, and for n sufficiently large, g is a d_n -Lipschitz extension of f_n , which is a contradiction. Likewise, if $L(a)$ is the strict derivative of \tilde{f} at a (with respect to X) as in (vii) then \tilde{f} is Lipschitz in $B_X(0, 2r)$ for some $r > 0$ and we obtain a contradiction. The same example also provides a contradiction with (vii).

Appendix A. Proof of Lemma 4.2

We derive Lemma 4.2 from a very similar statement that is proven in [EG, pp. 245–247] and summarized in [KZ].

Lemma A.1 (cf. [KZ] and [EG]). *Let $s \geq 1$. Then Lemma 4.2 holds true when (31) is replaced by*

$$r(x) = \frac{1}{20} \min(s, \text{dist}(x, F)). \quad (72)$$

Proof. Case $s = 1$. This case is exactly the one proven in [EG, pp. 245–247] and summarized in [KZ, Step 1 on p. 1031].

Case $s > 1$. The partition can be obtained from the previous case by scaling:

Let $F_* = \{x/s : x \in F\}$ and let $\{x_j\}_{j \in \mathbb{N}}$, $\{\phi_j\}_{j \in \mathbb{N}}$ be corresponding points and partition of unity from the previous case, that is, with the properties as in Lemma A.1 but with $s = 1$ and $F = F_*$. For $j \in \mathbb{N}$ and $x \in \mathbb{R}^n \setminus F$, let $\phi_j^*(x) = \phi_j(x/s)$ and $x_j^* = s x_j$. Then partition of unity $\{\phi_j^*\}_{j \in \mathbb{N}}$ on $\mathbb{R}^n \setminus F$ and points $\{x_j^*\}_{j \in \mathbb{N}}$ have all the required properties. \square

Proof of Lemma 4.2. We combine the partitions in such a way that each of them is used in a range of distances from F (with overlaps). We do that by multiplying each of them by a function v_{6^m} which is a member of a partition of unity that roughly depends only on distance from F . We can obtain $\{v_{6^m}\}_{m \in \mathbb{N}}$ either directly using the partitions at hand (as we do) or using the so called regularized distance.

³Namely that $(\tilde{f})'$ is continuous at a or that $L(a)$ is the strict derivative of \tilde{f} at a (even only with respect to $(\mathbb{R}^n \setminus F) \cup \{a\}$).

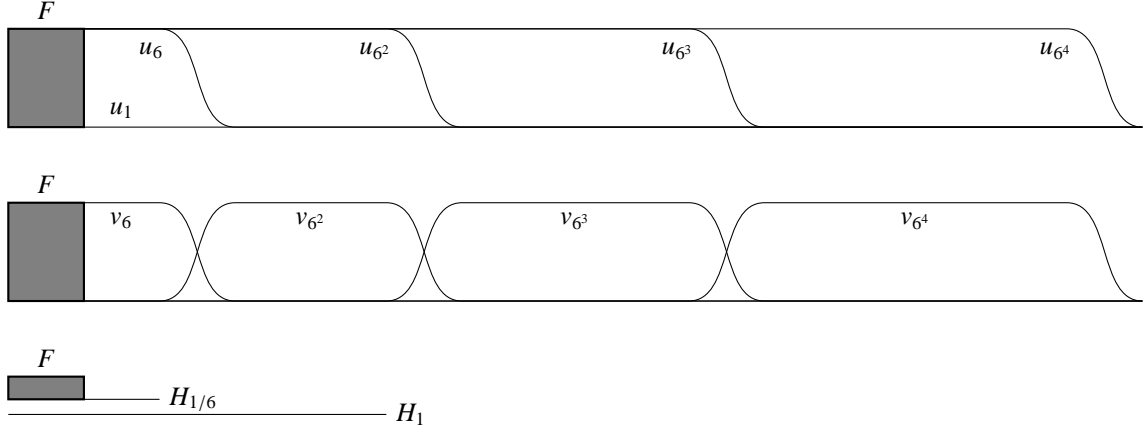


Figure 2: Schematic depiction of u_s , v_s and H_s (not to scale). Published with permission of © Jan Kolář 2016. All Rights Reserved.

For $s \geq 1$, let r_s , $\{x_{s,j}\}_{j \in \mathbb{N}}$ and $\{\phi_{s,j}\}_{j \in \mathbb{N}}$ denote the function from (72) corresponding to s , and the points and partition of unity from Lemma A.1. Thus $r_s(x) = (1/20) \min(s, \text{dist}(x, F))$. Let also $r(x) = (1/20) \text{dist}(x, F)$. Note that $0 \leq \phi_{s,j} \leq 1$ for every $s \geq 1$ and $j \in \mathbb{N}$.

For $d > 0$, denote $H_d = \{x : \text{dist}(x, F) \leq d\}$. Then, for every $x \in \mathbb{R}^n \setminus F$,

$$B(x, 10r(x)) \subset H_{(3/2)\text{dist}(x,F)} \setminus H_{(1/2)\text{dist}(x,F)} = H_{30r(x)} \setminus H_{10r(x)}. \quad (73)$$

Moreover, if $y \in B(x, 10r(x))$ then

$$B(y, 10r(y)) \subset H_{(3/2)\text{dist}(y,F)} \setminus H_{(1/2)\text{dist}(y,F)} \subset H_{(9/4)\text{dist}(x,F)} \setminus H_{(1/4)\text{dist}(x,F)} = H_{45r(x)} \setminus H_{5r(x)}. \quad (74)$$

Let $J_s = \{j \in \mathbb{N} : x_{s,j} \in H_{s/18}\}$ and $u_s = \sum_{j \in J_s} \phi_{s,j}$ for all $s \geq 1$. Also, let

$$\begin{aligned} G_s &= \bigcup \{B(x_{s,j}, 10r(x_{s,j})) : j \in J_s\} \subset H_{s/12}, \\ O_s &= \bigcup \{B(x_{s,j}, 10r(x_{s,j})) : j \in \mathbb{N} \setminus J_s\} \subset \mathbb{R}^n \setminus H_{s/36}. \end{aligned}$$

By (35) and (72), we have $\text{supp } u_s \subset G_s$ and $\text{supp}(1 - u_s) \subset O_s$. In particular, $\text{supp } u_{s/6} \subset H_{s/72} \subset H_{s/36}$ does not intersect $\text{supp}(1 - u_s)$ and

$$u_s u_{s/6} = u_{s/6} \quad \text{on } \mathbb{R}^n \setminus F. \quad (75)$$

Set (redefine) $u_1 = 0$ and for $s \geq 6$, let

$$v_s(x) = u_s(x) (1 - u_{s/6}(x)), \quad x \in \mathbb{R}^n \setminus F.$$

Then $\text{supp } v_s \subset G_s \cap O_{s/6} \subset H_{s/12} \setminus H_{s/216}$ for all $s > 6$ and $\text{supp } v_s \subset G_s \subset H_{s/12}$ for $s = 6$. Also, $\sum_{m \in \mathbb{N}} v_{6^m}(x) = 1$ for every $x \in \mathbb{R}^n \setminus F$ (see Figure 2). Indeed, on $\mathbb{R}^n \setminus F$,

$$1 = \sum_{m \in \mathbb{N}} u_{6^m} - u_{6^{m-1}} \stackrel{(75)}{=} \sum_{m \in \mathbb{N}} u_{6^m} - u_{6^m} u_{6^{m-1}} = \sum_{m \in \mathbb{N}} u_{6^m} (1 - u_{6^{m-1}}) = \sum_{m \in \mathbb{N}} v_{6^m}.$$

Let

$$\begin{aligned} M_s &= \{j \in \mathbb{N} : x_{s,j} \in H_{s/6} \setminus H_{s/324}\} \stackrel{(73)}{\supset} \{j \in \mathbb{N} : B(x_{s,j}, 10r(x_{s,j})) \cap (H_{s/12} \setminus H_{s/216}) \neq \emptyset\} & \text{for } s > 6, \\ M_s &= \{j \in \mathbb{N} : x_{s,j} \in H_{s/6}\} \stackrel{(73)}{\supset} \{j \in \mathbb{N} : B(x_{s,j}, 10r(x_{s,j})) \cap H_{s/12} \neq \emptyset\} & \text{for } s = 6, \end{aligned}$$

and

$$\mathcal{J}_{x,s} = \{j \in M_s : B(x, 10r(x)) \cap B(x_{s,j}, 10r(x_{s,j})) \neq \emptyset\}.$$

If $\mathcal{J}_{x,s} \neq \emptyset$ then $x \in H_{s/2} \setminus H_{s/972}$ if $s > 6$ and $x \in H_{s/2}$ if $s = 6$. Thus, for a fixed $x \in \mathbb{R}^n \setminus F$, there are at most four different $s = 6^m$ such that $m \in \mathbb{N}$ and $\mathcal{J}_{x,6^m} \neq \emptyset$. Moreover, $r(x) = r_s(x)$ for all $x \in H_s$, in particular $r(x_{s,j}) = r_s(x_{s,j})$ for all $j \in M_s$, and $r(x) = r_s(x)$ whenever $\mathcal{J}_{x,s} \neq \emptyset$. Then also $\text{Card } \mathcal{J}_{x,s} \leq C_1$ by (32).

Consider

$$\left\{x_{6^m,j}\right\}_{m \in \mathbb{N}, j \in M_{6^m}} \quad \text{and} \quad \left\{v_{6^m}(x) \phi_{6^m,j}(x)\right\}_{m \in \mathbb{N}, j \in M_{6^m}}.$$

Note that the condition $j \in M_{6^m}$ (compared to $j \in \mathbb{N}$) removes only (some of) the elements where $v_{6^m}(x) \phi_{6^m,j}(x)$ is the zero function. Therefore $\sum_{m \in \mathbb{N}} \sum_{j \in M_{6^m}} v_{6^m}(x) \phi_{6^m,j}(x) = \sum_{m \in \mathbb{N}} \sum_{j \in \mathbb{N}} v_{6^m}(x) \phi_{6^m,j}(x) = \sum_{m \in \mathbb{N}} v_{6^m}(x) = 1$ for $x \in \mathbb{R}^n \setminus F$. Obviously, $\text{supp } v_{6^m} \phi_{6^m,j} \subset \text{supp } \phi_{6^m,j} \subset B(x_{6^m,j}, 10r_s(x_{6^m,j})) = B(x_{6^m,j}, 10r(x_{6^m,j}))$ whenever $j \in M_{6^m}$.

Let η be a bijection $\eta: \mathbb{N} \rightarrow \{(6^m, j) : m \in \mathbb{N}, j \in M_{6^m}\}$. Let $\phi_k^\#(x) = v_s(x) \phi_{s,j}(x)$ and $x_k^\#(x) = x_{s,j}$ for $x \in \mathbb{R}^n \setminus F$, where $(s, j) = \eta(k)$.

We claim that $\{\phi_k^\#\}_{k \in \mathbb{N}}$ is a partition of unity in $\mathbb{R}^n \setminus F$ with the required properties but with C_1, C_2 replaced by $C_1^* := 4C_1, C_2^* := 3C_1C_2$. To show that, fix $x \in \mathbb{R}^n \setminus F$. Since $r_s(x) \leq r(x)$ for every $s \geq 1$, we have

$$\mathcal{J}_x \subset \{k \in \mathbb{N} : j \in \mathcal{J}_{x,s} \text{ where } (s, j) = \eta(k)\}.$$

Recall that there are at most four different $s = 6^m$ ($m \in \mathbb{N}$) such that $\mathcal{J}_{x,s} \neq \emptyset$. And, for each such s , $\text{Card } \mathcal{J}_{x,s} \leq C_1$. Hence $\text{Card } \mathcal{J}_x \leq 4C_1$. Furthermore, for every $k \in \mathbb{N}$ and $(s, j) = \eta(k)$,

$$\left|(\phi_k^\#)'(x)\right| \leq \left|\phi'_{s,j}(x)\right| + \left|u'_s(x)\right| + \left|u'_{s/6}(x)\right| \leq C_2/r(x) + 2C_1C_2/r(x) \leq 3C_1C_2/r(x). \quad \square$$

Appendix B. Extensions from special closed sets $F \subset \mathbb{R}^n$

Forthcoming paper [Ko] contains generalizations of results of [KZ, Section 4] that avoid the assumption of L being Baire one or continuous and replace them by requirements on the set F . We give here some of the results with concise and self-contained proofs. For a stronger theorem with lengthy proof and a number of other corollaries, see [Ko].

Recall from [KZ] that, for $F \subset \mathbb{R}^n$ and $x \in \mathbb{R}^n$,

$$\text{Tan}(F, x) \subset \text{Ptg}(F, x), \quad (76)$$

where

$$\text{Tan}(F, x) = \{v \in \mathbb{R}^n : \text{there are } x_k \in F \text{ and } \alpha_k \in [0, \infty) \text{ (for all } k \in \mathbb{N}) \text{ such that } x_k \rightarrow x \text{ and } \alpha_k(x_k - x) \rightarrow v\}$$

and

$$\text{Ptg}(F, x) = \{v \in \mathbb{R}^n : \text{there are } x_k, y_k \in F \text{ and } \alpha_k \in \mathbb{R} \text{ (for all } k \in \mathbb{N}) \text{ such that } x_k \rightarrow x, y_k \rightarrow x \text{ and } \alpha_k(y_k - x_k) \rightarrow v\}.$$

The sets $\text{Tan}(F, x)$ and $\text{Ptg}(F, x)$ are called the *contingent cone* (sometimes also the *tangent cone*) of F at x and the *paratingent cone* of F at x , respectively. Let $\text{der } F$ denote the set of accumulation points of F .

We need the following generalization of [KZ, Lemma 4.9] to vector-valued functions:

Lemma B.1. *Let Y be a normed linear space. Let $F \subset \mathbb{R}^n$, $x \in F \cap \text{der } F$, $f: F \rightarrow Y$ and let $L \in \mathcal{L}(\mathbb{R}^n, Y)$ be a strict derivative of f at x . Let $v_1, \dots, v_n \in \text{Ptg}(F, x)$ be unit vectors and $|\det(v_1, \dots, v_n)| > d > 0$. Then*

$$\|L\|_{\mathcal{L}(\mathbb{R}^n, Y)} \leq \limsup_{\substack{y \rightarrow x, z \rightarrow x, \\ y, z \in F, y \neq z}} \frac{n \|f(y) - f(z)\|_Y}{d |y - z|}. \quad (77)$$

Proof. For the case $Y = \mathbb{R}$, see [KZ, Lemma 4.9] where (77) is presented in the form

$$\limsup_{\substack{y \rightarrow x, z \rightarrow x, \\ y, z \in F, y \neq z}} \frac{\|f(y) - f(z)\|_Y}{|y - z|} > \frac{d\varepsilon}{n}$$

whenever $\|L\|_{\mathcal{L}(\mathbb{R}^n, Y)} > \varepsilon > 0$. We only need to treat the vector case. If $\|L\|_{\mathcal{L}(\mathbb{R}^n, Y)} > \varepsilon > 0$ then there obviously exists $\phi \in Y^*$ such that $\|\phi\|_{Y^*} = 1$ and $\|\phi L\|_{\mathcal{L}(\mathbb{R}^n, \mathbb{R})} > \varepsilon$. Now it is enough to apply [KZ, Lemma 4.9] to the real function $f_0(y) = \phi(f(y))$, $y \in F$. \square

From Lemma B.1, the following statement inspired by [ALP, Corollary 2] (see also [KZ, Lemma U]) immediately follows:

Lemma B.2. *Let Y be a normed linear space, $F \subset \mathbb{R}^n$ and $x \in F \cap \text{der } F$. If $f: F \rightarrow Y$ is relatively strictly differentiable at x and $\text{Ptg}(F, x)$ spans \mathbb{R}^n , then the relative strict derivative of f at x is determined uniquely.*

Proof. If $\text{Ptg}(F, x)$ spans \mathbb{R}^n then $\text{span}\{v_1, \dots, v_n\} = \mathbb{R}^n$ for some vectors $v_1, \dots, v_n \in \text{Ptg}(F, x)$. We can assume that v_1, \dots, v_n are unit vectors. Choose any $0 < d < |\det(v_1, \dots, v_n)|$. Assume that $f: F \rightarrow Y$ is strictly differentiable at x and L_1, L_2 are two distinct strict derivatives of f at x . Let $f_0(y) = f(y) - L_1(y)$ for $y \in F$ and $L_0 = L_2 - L_1$. Then $L_0 \neq 0$ is a strict derivative of f_0 at x . By Lemma B.1, (77) holds true for f_0 and L_0 with $\|L_0\|_{\mathcal{L}(\mathbb{R}^n, Y)} > 0$. However, this contradicts the fact that 0 is also a strict derivative of f_0 at x . \square

Now, we can formulate another corollary to Theorem 4.1. The following result is a generalization of [KZ, Proposition 4.10] (we allow vector-valued mappings and also replace $\text{Tan}(F, x)$ by a larger set $\text{Ptg}(F, x)$).

Proposition B.3. *Let $F \subset \mathbb{R}^n$ be a nonempty closed set such that $\text{Ptg}(F, x)$ spans \mathbb{R}^n for every $x \in \text{der } F$.⁴ Let Y be a normed linear space and $f: F \rightarrow Y$ a function (relatively) strictly differentiable at every $x \in \text{der } F$. Then there exists a differentiable extension of f defined on \mathbb{R}^n .*

Proof. Let $L: F \rightarrow \mathcal{L}(\mathbb{R}^n, Y)$ be a (relative) strict derivative of f on F . Note that L is uniquely determined on $\text{der } F$ by Lemma B.2. To finish the proof with the help of Theorem 4.1, we only need to prove that L is a Baire one function on F . We follow [KZ] by letting

$$F_m = \left\{ x \in \text{der } F : \sup \{ \det(v_1, \dots, v_n) : v_1, \dots, v_n \text{ unit vectors from } \text{Ptg}(F, x) \} \geq \frac{1}{m} \right\} \quad \text{for } m \in \mathbb{N}.$$

By the proof of [KZ, Proposition 4.10], F_m is closed for every $m \in \mathbb{N}$ and $\bigcup_{m \in \mathbb{N}} F_m = \text{der } F$ (the fact that $\text{Ptg}(F, x)$ spans \mathbb{R}^n for every $x \in \text{der } F$ is used). Authors of [KZ] also prove that L is continuous on F_m for every $m \in \mathbb{N}$, and the same proof can be applied unchanged to vector-valued functions f and L , with the help of Lemma B.1 instead of [KZ, Lemma 4.9].

Now, it is easy to deduce that L is F_σ -measurable on F and [KZ] deduce that L is Baire one. For this step with vector-valued functions, we need to refer to [Ka, Corollary 4.13]. Alternatively, we construct a sequence of continuous functions $\{L_m\}_{m \in \mathbb{N}}$ that converges point-wise to L . Observe that $F_{m+1} \supset F_m$ for every $m \in \mathbb{N}$ and let H_n ($n \in \mathbb{N}$) be an increasing sequence of finite sets such that $F \setminus \text{der } F \subset \bigcup_{m \in \mathbb{N}} H_m \subset F$. By Dugundji's extension theorem or our Theorem 1.1(ii)(v),⁵ we let $L_m: F \rightarrow \mathcal{L}(\mathbb{R}^n, Y)$ be a continuous extension of the (continuous) function $L|_{F_m \cup H_m}$. Obviously, L is the point-wise limit of L_m . \square

Once we assume strict differentiability, is it natural to ask about stronger conclusions with strict differentiability or C^1 smoothness. Of course, we need an additional assumption that enforces continuity of the strict derivative.

Proposition B.4. *Under the assumptions of Proposition B.3, there exists a differentiable extension $\tilde{f}: \mathbb{R}^n \rightarrow Y$ of f such that \tilde{f} is strictly differentiable at x (with respect to \mathbb{R}^n) and the derivative of \tilde{f} is continuous at x (with respect to \mathbb{R}^n) for all*

- (a) $x \in \mathbb{R}^n \setminus \text{der } F$ and
- (b) $x \in \text{der } F$ where the (unique) relative strict derivative of f (with respect to F) is continuous with respect to $\text{der } F$.

Proof. Let $L_0: F \rightarrow \mathcal{L}(\mathbb{R}^n, Y)$ be a strict derivative of f on F (recall that at isolated points, any element of $\mathcal{L}(\mathbb{R}^n, Y)$ is a strict derivative of f with respect to F). By the proof of Proposition B.3, L_0 is a Baire one function on F . So is the restriction $L_0|_{\text{der } F}$ to the closed set $\text{der } F$. Using Theorem 1.6 with $L = L_0|_{\text{der } F}$, we obtain $A: \mathbb{R}^n \setminus \text{der } F \rightarrow \mathcal{L}(\mathbb{R}^n, Y)$ such that the “union of functions” $L_1 := L_0|_{\text{der } F} \cup A$ is an extension of $L_0|_{\text{der } F}$ that is Baire one on \mathbb{R}^n and continuous at every point of $\mathbb{R}^n \setminus \text{der } F$ and at every point of $\text{der } F$ where $L_0|_{\text{der } F}$ is continuous. Let $L = L_1|_F$. The extension \tilde{f} of f provided by Theorem 4.1 (with special regard to (vi)) is strictly differentiable at all points promised. Let $x \in F$ be as in (a) or (b). Then the continuity of the derivative of \tilde{f} at x with respect to $(\mathbb{R}^n \setminus F) \cup \{x\}$ comes from Theorem 4.1(vi). This combines with the continuity of the relative strict derivative of f at x with respect to F assumed in (b). \square

⁴Of course, $\text{Ptg}(F, x)$ can be replaced by $\text{Tan}(F, x)$, which is smaller but probably better known.

⁵We use Theorem 1.1 with $f_{\text{Theorem 1.1}} := L|_{F_m \cup H_m}$, $L_{\text{Theorem 1.1}} := 0$, $Y_{\text{Theorem 1.1}} := \mathcal{L}(\mathbb{R}^n, Y)$ and then we let $L_m = (\tilde{f}_{\text{Theorem 1.1}})|_F$.

The continuity assumption cannot be removed from Proposition B.4(b) as can be seen from [KZ, Example 4.14]. However, it can be replaced by an assumption on F , see [Ko].

A particular corollary under the assumptions of Proposition B.4 is the following statement: if the (unique) relative strict derivative of f on $\operatorname{der} F$ is assumed to be continuous on the whole $\operatorname{der} F$ then there exists a C^1 extension of f to \mathbb{R}^n . However, this follows from Whitney's theorem, even without the assumptions on $\operatorname{Ptg}(F, x)$. (The extension formula [W, (11.1)] of Whitney works well for vector-valued functions.)

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